You may bring one 8.5x11" sheet of paper on both sides of which you have written whatever you like. Bring no other notes nor text nor calculator nor telephone. Work out solutions to as many of the following problems as you can on scratch paper, which will be supplied, and then rewrite neatly as much of each solution as you want graded on a fresh sheet with your NAME and the problem's number PRINTED on it when you hand it in. Partial solutions earn much less credit than complete solutions. They will show up on `www.cs.berkeley.edu/~wkahan/Math185.`

Problem 0: " For any integer $n \ge 2$ we find $\int_C z^{-n}\,dz = 0$ around every simple closed curve C, no matter whether it encloses $z = 0$, so Morera's theorem implies z^{-n} is holomorphic everywhere in the finite complex plane." Where is the flaw in this reasoning? 5 min.

Solution 0: Morera's theorem asserts that $f(z)$ is holomorphic in any open domain where f is ***continuous*** and every such $\int_C f(z)\cdot dz = 0$. The problem's integrand $f(z) = 1/z^n$ is so far from continuous at $z = 0$ as to invalidate the problem's quoted assertion whenever C passes through 0. (In this case, if C is smooth (continuously varying tangent) except for a corner at 0 where the interior angle is $\pi/(n-1)$, the integral turns out to be infinite, not zero.)

Problem 1: Exhibit an example of a Taylor series that converges everywhere inside and on the boundary of its finite circle of convergence but (of course) nowhere outside. 5 min.

Solution 1: $\sum_{n\ge1} z^n/n^2$, or $(1-z)\cdot\log(1-z) = -z + \sum_{n\ge1} z^{n+1}/((n+1)n)$, or … . The *Ratio Test* confirms convergence inside and divergence outside the unit circle on which convergence is dominated by a telescoping series of positive terms.

Problem 2: Suppose $U(x, y)$ is a non-constant harmonic function whose value is a constant everywhere on some simple smooth closed curve (not just an arc or one point) strictly inside U's domain. How does this constrain U's domain, and why? Exhibit an explicit example $U(x, y)$ and determine one of its harmonic conjugates $V(x, y)$, and describe V's domain. 10 min.

Solution 2: Because non-constant harmonic functions cannot take maximum nor minimum values at interior points of their domains, the curve in question must enclose some boundary point(s) of U's domain; ***its domain cannot be simply connected***. $U(x, y) := \log(x^2 + y^2)$ is an example; it takes constant values on circles in a domain punctured by U's singularity at their common center, the origin. U's harmonic conjugates are $V(x, y) = 2\cdot\mathrm{Arg}(x + \imath y) + \text{constant}$, multi-valued unless the (x, y)-plane is cut by any simple (non self-intersecting) curve from the origin to infinity; then the cut plane is V's domain. If the cut slits the negative x-axis, one harmonic conjugate of U becomes $V(x, y) = 4\cdot\arctan(y/(x + \sqrt{x^2 + y^2}))$ for $y \ne 0$ or $x > 0$.

A "closed curve" through ∞ is ruled out because ∞ cannot be "strictly inside" U's domain in the (x, y)-plane.

Problem 3: Let $g(w) := (w^2 + 3)/(w^2 - 3w)$. Let O be the unit circle traversed counter-clockwise. Obtain for $f(z) := \int_O g(w)\,dw/(w-z)$ two explicit *closed-form* expressions (with no integral sign, no infinite series), one valid when z is inside O , the other valid outside.

10 min.

Solution 3: $f(z) = 2\pi\imath\cdot(1 + 4/(z-3))$ if $|z| < 1$. $f(z) = 2\pi\imath/z$ if $|z| > 1$. Here is why:

The partial fraction expansion $g(w) = 1 + 4/(w-3) - 1/w$ leads to a partial fraction expansion

$$g(w)/(w-z) = 1/(w-z) + 4\big(1/(w-z) - 1/(w-3)\big)/(z-3) - \big(1/(w-z) - 1/w\big)/z \quad \text{provided } z \neq 0 .$$

Then the integral $f(z)$ turns out to be the change, as w traverses O , in

$$\mathrm{Log}(w-z) + 4\big(\mathrm{Log}(w-z) - \mathrm{Log}(w-3)\big)/(z-3) - \big(\mathrm{Log}(w-z) - \mathrm{Log}(w)\big)/z .$$

The same solution can be computed from residues. The singularity at $z = 0$ removes itself.

Problem 4: Which if any of the following 4 analytic functions are equal *almost everywhere* in the complex z-plane, and why? (Here $\sqrt{\ldots}$ is the *Principal Square Root* with $\mathrm{Re}(\sqrt{\ldots}) \geq 0$.)

$$A(z) := \sqrt{z-1}\cdot\sqrt{z+1} ; \quad B(z) := \sqrt{(z^2-1)} ; \quad C(z) := (z+1)\cdot\sqrt{(z-1)/(z+1)} ; \quad D(z) := \imath\cdot\sqrt{(1-z^2)} .$$

20 min.

Answer 4: Only $A(z) = C(z)$ everywhere after the 0/0-singularity of C is removed by setting $C(-1) := 0$. They have the same slit $-1 < z < 1$. The other functions have different slits across which they reverse sign. B 's slit is cross-shaped including the imaginary axis plus the real line segment $-1 < z < 1$. And D has two slits on the real axis: $z > 1$ and $z < -1$. On the ray $z > 1$ we find $A(z) = B(z) = C(z)$ so the *Monodromy/Identity Theorem* makes them agree on the largest open domain that includes the ray and inside which domain all three functions are holomorphic. $A = B = C$ if $\mathrm{Re}(z) > 0$ but $A = C = -B$ away from B 's slit where $\mathrm{Re}(z) < 0$.

Problem 5: Suppose $\mathbb{F}$ and $\mathbb{G}$ are disjoint open Domains whose boundaries abut along an open smooth curve C ; and suppose f is holomorphic inside $\mathbb{F}$ and continuous over $C\cup\mathbb{F}$, g is holomorphic inside $\mathbb{G}$ and continuous over $C\cup\mathbb{G}$, and $f = g$ along C . Prove *Painlevé's Theorem*: The function h defined by $h := f$ in $C\cup\mathbb{F}$ and $h := g$ in $C\cup\mathbb{G}$ is holomorphic inside $\mathbb{F}\cup C\cup\mathbb{G}$.

20 min.

Solution 5: *Morera*'s theorem will say our continuous h is holomorphic inside its domain if we prove $\int_O h(z)dz = 0$ around every loop O inside that domain (even if loops are restricted to triangles or to rectangles). We cope with loops that do not cross C by applying the *Cauchy-Goursat* theorem to either f in $C\cup\mathbb{F}$ or g in $C\cup\mathbb{G}$, using that version of this theorem that allows parts of a loop to coincide with boundary arcs provided the integrand stays continuous approaching and along each arc. Any loop O that does cross C is cut by C , as the / cuts Ø , into at least two smaller loops, each either in $C\cup\mathbb{F}$ or in $C\cup\mathbb{G}$ so the integral around it vanishes; the zero sum of the integrals around the smaller loops equals the integral around O because the contributions from cuts, each traversed twice in opposite directions, cancel out. End of proof. **Note**: Differentiability of h along or across C cannot be presumed at the outset.

Problem 6: $M(z) := (\alpha z + \beta)/(\gamma z + \delta)$ has real coefficients $\alpha, \beta, \gamma, \delta$ with $\mu := \alpha\delta - \beta\gamma \neq 0$. Explain how $\text{sign}(\mu)$ determines the region to which M maps the half-plane where $\text{Im}(z) > 0$.

20 min.

Solution 6: $M(x + \imath y) = ((\alpha x+\beta)(\gamma x+\delta) + \alpha\gamma y^2 + \imath\mu y)/|\gamma x+\delta + \imath\gamma y|^2$, implying that $\text{sign}(\text{Im}(M(z))) = \text{sign}(\mu)\cdot\text{sign}(\text{Im}(z))$. Therefore M maps the upper half-plane $\text{Im}(z) > 0$ one-to-one onto itself if $\mu > 0$, but onto the lower half-plane if $\mu < 0$. M maps the extended real axis to itself with order preserved if $\mu > 0$, reversed if $\mu < 0$, because $M'(z) = \mu/(\gamma z + \delta)^2$; this also implies that the upper hemisphere of the *Riemann Sphere* $\mathbb{S}$, lying to the left of the real axis' image ($\mathbb{S}$'s equator) when traversed in increasing order, is mapped to the upper hemisphere when $\mu > 0$, to the lower when $\mu < 0$, thus corroborating the previous sentence.

Problem 7: Integer $n \geq 3$. How many zeros of $z^n - 2\cdot z + 1$ lie strictly inside the unit circle, and why?

30 min.

Solution 7: Just one zero lies strictly inside the unit circle. One way to prove this first factors $f(z) := z^n - 2z + 1 = (z-1)(z^{n-1} + z^{n-2} + z^{n-3} + \dots + z - 1)$. As z runs from 0 to 1 the second factor runs up from $-1 < 0$ to $n-2 > 0$, so it has at least one zero strictly inside the unit circle; and f has another zero at $z = 1$ on the unit circle. The zeros of $f_\mu(z) := z^n - 2\mu\cdot z + 1$ are continuous functions of μ and approach the zeros of f as μ decreases towards 1 through real values. When $|z| = 1 < \mu$ we find $|2\mu z| = 2\mu > 2 \geq |z^n + 1|$, and then f_μ has no zero on the unit circle and, by Rouché's theorem, just as many zeros inside the unit circle as $2\mu z$ has, namely one zero. As μ decreases to 1 none of the $n-1$ zeros of f_μ outside the unit circle can jump across it into its interior, so the one zero inside stays strictly inside while one of the outside zeros z moves to $z = 1$ on the unit circle.

Another way to prove that just one zero of f lies inside the unit circle starts by observing that $f(1/2) > 0$, $f(1) = 0$ and $f'(1) = n-2 > 0$. Consequently f has at least one real zero strictly between 1/2 and 1 ; call the largest such zero r so that $f(x) < 0$ whenever $r < x < 1$. Let R be chosen strictly between 1 and the largest of the magnitudes of the (perhaps complex) zeros of f strictly inside the unit circle; evidently $1/2 < r < R < 1$, and so $f(R) < 0$. On the circle $|z| = R$ we find $|-f(z)| \geq |2z - 1| - |z^n| \geq 2|z| - 1 - |z|^n = -f(R) > 0$, so f has no zero on that circle and, by Rouché's theorem, just as many zeros strictly inside that circle as $2z-1$ has, namely one zero. (This zero of f is r .) End of proofs.

Problem 8: The coefficients in the series $f(z) = \sum_{n\geq 0} t_n z^n/n!$ are determined by a recurrence

$$t_0 := 0 ; \quad t_1 := 1 ; \quad t_{k+1} := \sum_{0 \leq j \leq k} {}^kC_j\, t_j\, t_{k-j} \quad \text{for } k = 1, 2, 3, 4, \dots$$

wherein the combinatorial coefficient ${}^kC_j := k!/(j!\cdot(k-j)!)$. By finding a differential equation $f' = 1 + \dots$ that f must satisfy, identify $f(z)$ and determine its series' radius of convergence.

30 min.

Solution 8: Radius = $\pi/2$. Although all the t_n's are integers and easy to compute (and every $t_{2n} = 0$), the problem is easier understood in terms of the coefficients $f_n := t_n/n!$ of the series $f(z) = \sum_{n\ge0} f_n z^n$. Within its circle of convergence the derivative $f'(z) = \sum_{n\ge0} n\cdot f_n z^{n-1}$; it has $(n+1)\cdot f_{n+1}$ as the coefficient of z^n . For all $n > 0$, according to the given recurrence, $(n+1)\cdot f_{n+1} = \sum_{0\le j\le n} f_j f_{n-j}$ turns out also to be the coefficient of z^n in $f(z)^2$. It soon follows that $f' = 1 + f^2$, and consequently $f(z) = \tan(z)$. Its singularities nearest $z = 0$ are at $z = \pm\pi/2$, so the series' radius of convergence is $\pi/2$.

Problem 9: Which, if any, functions $f(z)$ are holomorphic for all $|z| \le 1$ and have infinitely many zeros z satisfying $|z| < 1$, and why? Where are the zeros of $e^{z/(z-1)} - 1$, and why do they not contradict what you just claimed about functions $f(z)$? 20 min.

Solution 9: Only $f(z) = 0$ can be holomorphic in the closed unit disk and have infinitely many zeros strictly inside it. They must have at least one accumulation-point z_o in the closed unit disk, perhaps on the unit circle. Since f is holomorphic on the closed unit disk (and therefore slightly outside it too), f is continuous at z_o and thus $f(z_o) = 0$. From its Taylor series expansion we know that a nonzero holomorphic function can have only isolated zeros, but z_o is not one of those, so f must be zero everywhere.

However $g(z) := e^{z/(z-1)} - 1$ has infinitely many zeros $z = 2n\pi\imath/(1 + 2n\pi\imath)$ for each of which $z/(1-z) = 2n\pi\imath$ and n is an integer. Evidently they all satisfy $|z| < 1$; in fact they all lie on the part of the circle $|z - 1/2| = 1/2$ that lies strictly inside the unit circle. The zeros' limit-point $z = 1$, corresponding to $n = \infty$, is not a zero of $g(z)$ but an essential singularity, so $g(z)$ is not holomorphic on ***all*** of the closed unit disk, unlike what was assumed about $f(z)$.

Problem 10: Suppose f and g are holomorphic inside and on a smooth simple closed curve C . Explain why $\int_{\text{around } C} \overline{f(z)}\cdot g(z)\cdot dz = 2\imath\cdot\iint_{\text{inside } C} \overline{f'(x+\imath y)}\cdot g(x+\imath y)\cdot dx\cdot dy$. 20 min.

Solution 10: Recall *Green's Theorem in the Plane*:

> $\int_{\partial R} (P\cdot dx + Q\cdot dy) = \iint_R (\partial Q/\partial x - \partial P/\partial y)\, dx\, dy$ wherein $P(x, y)$ and $Q(x, y)$ are continuously differentiable functions, and R is a plane region whose boundary ∂R is a piecewise smooth closed curve traversed during the first integration in a direction that puts the interior of R on the left.

Here $Q/\imath = P = \overline{f}\cdot g$. Now, $\partial g(x+\imath y)/\partial x = g'(x+\imath y)$ and $\partial g(x+\imath y)/\partial y = g'(x+\imath y)\cdot\imath$, *etc.*, so $\partial Q/\partial x - \partial P/\partial y = \imath\cdot(\overline{f'}\cdot g + \overline{f}\cdot g') - (\overline{f'\cdot\imath}\cdot g + \overline{f}\cdot g'\cdot\imath) = 2\imath\cdot\overline{f'}\cdot g$ as claimed. Explicitly splitting f and g into their respective real and imaginary parts and then invoking the *Cauchy-Riemann* equations merely prolongs the algebraic agony.

Problem 11: Each instance f below is an *Entire* function (holomorphic throughout the finite complex plane) whose properties are in question:

(a) What can be deduced about f if its value is never real?
(b) What can be deduced about f if its value is never a positive real number?
(c) What can be deduced about f if $|f| > 2$ everywhere?
(d) What can be deduced about f if its value is never 0 ?
(e) What can be deduced about f if some given entire function g has $|g| \ge |f|$ everywhere?
(f) What can be deduced about f if $|f(z)| < 7 + 3|z|^4$ for all z ?

30 min.

Solution 11:

(a) What can be deduced about f if its value is never real? **Solution:** f = imaginary constant because $\mathrm{Im}(f)$ cannot vanish, so $\mathrm{Im}(f) > 0$ (say), and then $|\exp(\imath \cdot f)| \le 1$; but then $\exp(\imath f)$ is a bounded entire function which, by *Liouville's Theorem*, must be constant.

(b) What can be deduced about f if its value is never a positive real number? **Solution:** Again f = constant because the entire function $\exp(-\sqrt{1-f})$ is a constant of magnitude ≤ 1 .

(c) What can be deduced about f if $|f| > 2$ everywhere? **Solution**: Again f = constant because f never vanishes, so $1/f$ is an entire function with $|1/f| < 1/2$, so constant.

(d) What can be deduced about f if its value is never 0 ? **Solution:** Now $f = \exp(g)$ for an entire function $g := \int (f'/f)\cdot dz$ since $(f\cdot\exp(-g))' = (f' - f\cdot g')\cdot\exp(-g) = 0$. Although g(z) is a *branch* of $\mathrm{Log}(f(z))$, writing " $g = \log(f)$ " would be a mistake because any single-valued log(…) must be discontinuous across a slit which the values of f would have to avoid. For example, $f(z) := \exp(z)$ is an entire function that never vanishes, but every single-valued branch of Log(…) yields a function $\log(f(z)) = \log(\exp(z))$ that jumps at places where $g(z) = z$ does not jump; for the *Principal* branch of log those places constitute the horizontal lines on which $\mathrm{Im}(z)$ is an odd integer multiple of π .

(e) What can be deduced about f if some given entire function g has $|g| \ge |f|$ everywhere? **Solution:** $f = c\cdot g$ for some constant c with $|c| \le 1$. This is so because $|f/g| \le 1$ everywhere except perhaps at the zeros of g ; but these are removable singularities of f/g , according to Riemann's boundedness criterion for removable singularities. After their removal f/g turns into a bounded entire function which, by Liouville's theorem, must be constant.

(f) What can be deduced about f if $|f(z)| < 7 + 3|z|^4$ for all z ? **Solution:** f is a polynomial of degree ≤ 4 with | coefficient of z^4 | ≤ 3 and with | constant term $f(0)$ | < 7 . All this is so because subtracting off the first 4 terms in the Maclaurin series for $f(z)$ and then dividing by z^4 yields another entire function g with $|g| < 3.5$, say, for all $|z|$ big enough; Liouville's th'm implies g = constant = (coefficient of z^4) = $\lim_{z\to\infty} f(z)/z^4$.