A holomorphic function $f(z)$ is declared to satisfy the equations

$$f(z) - \ln(1 + f(z)) = z^2/2\ , \quad f(0) := 0 \quad \text{and} \quad f'(0) := 1\ .$$

Here " $\ln(1 + f(z))$ " is the *Principal Logarithm* with $-\pi \le \mathrm{Imag}(\ln(\dots)) \le \pi$. Other equations satisfied by $f(z)$ are

$$f(z) = \exp\big(f(z) - z^2/2\big) - 1 \qquad \text{and} \qquad f'(z) = z\cdot(1 + 1/f(z))\ ;$$

the latter equation is a *Singular* differential equation with two *Regular* solutions from which the one selected here satisfies the initial conditions $f(0) := 0$ and $f'(0) := +1$. The selected solution has a *Taylor Series* expansion about the point $z = 0$ obtainable by repeated implicit differentiation of the equation $f\cdot f' = z\cdot f + z$ and subsequent substitution of $z = 0$ and $f = 0$:

$$f\cdot f'' + (f')^2 = z\cdot f' + f + 1 \qquad f'(0) = +1 \qquad (-1 \text{ is rejected})\ ;$$
$$f\cdot f''' + 3f'\cdot f'' = z\cdot f'' + 2f' \qquad f''(0) = 2/3\ ;$$
$$f\cdot f'''' + 4f'\cdot f''' + 3(f'')^2 = z\cdot f''' + 3f'' \qquad f'''(0) = 1/6\ ; \qquad \text{and so on} \dots .$$

Thus $f(z) = z + z^2/3 + z^3/36 - z^4/270 + z^5/4320 + z^6/17010 - 139z^7/5443200 + z^8/204120 - \dots$ in the intersection of the domain of $f(z)$ with the circular disk inside which this series converges.

What is the radius r of that disk? Why is $r = 2\sqrt{\pi} \approx 3.5449$?

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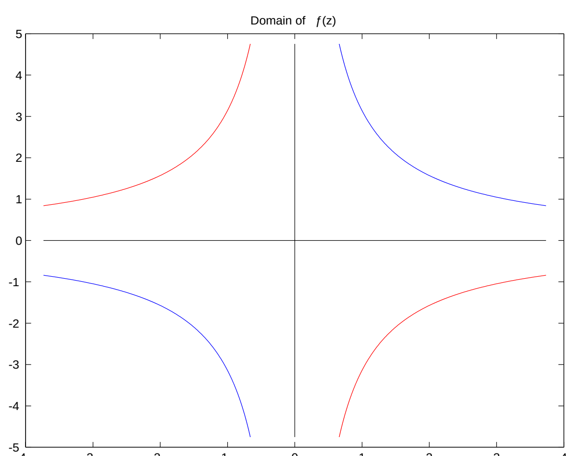
The *Cauchy-Hadamard* formula $r = \liminf_{n\to\infty} |a_n|^{-1/n}$ requires a formula for the coefficients a_n of z^n in the series, but I know no such formula. Numerical computation of the coefficients does not help much because they vary so irregularly that convergence to r is slow; for instance,

$$|a_{44}|^{-1/44} = (4.69459_{10}{-27})^{-1/44} = 3.966$$
$$|a_{45}|^{-1/45} = (4.41717_{10}{-29})^{-1/45} = 4.267$$
$$|a_{46}|^{-1/46} = (3.32176_{10}{-28})^{-1/46} = 3.957$$
$$|a_{47}|^{-1/47} = (1.31482_{10}{-28})^{-1/47} = 3.919$$
$$|a_{48}|^{-1/44} = (2.59963_{10}{-29})^{-1/48} = 3.940$$

The series' radius of convergence is the distance r from $z = 0$ to the nearest singularity of a function $F(z)$ that agrees with $f(z)$ on the intersection of their domains, but the domain of $F(z)$ extends as far as analytic continuation can take it beyond the domain $\mathbb{S}$ of $f(z)$, which is the region swept out by $z = \pm\sqrt{\big(2(w - \ln(1+w))\big)}$ as w sweeps through the whole complex plane. Domain $\mathbb{S} = -\mathbb{S} = \overline{\mathbb{S}}$ lies inside a four-pointed star, the image of logarithm's slit where $w < -1$: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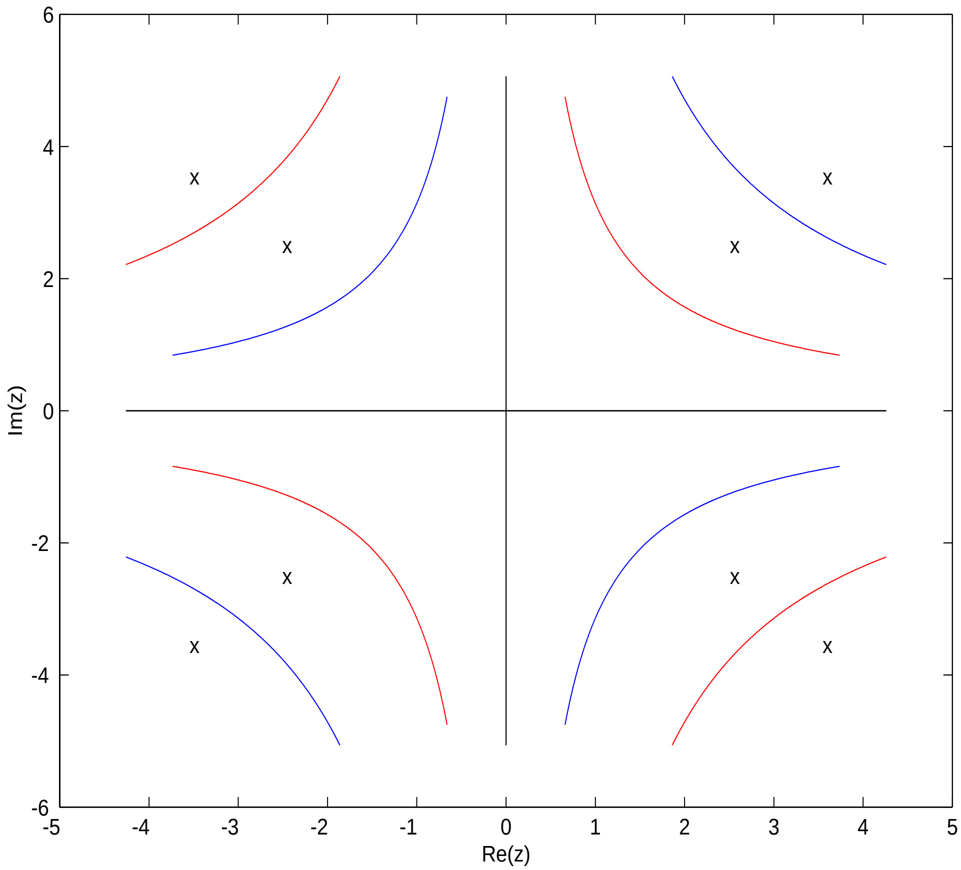
If $z = \pm\sqrt{(2(w - \ln(1+w)))}$ is to cross from inside $\mathbb{S}$ to outside this star-shaped region, w must cross the logarithm's slit, crossing the line where $w < -1$ and $\ln(1+w)$ jumps by $\pm 2\imath\pi$; but for the sake of z's continuity " $\ln(1+w)$ " must be replaced by an adjacent branch of the logarithm function. In other words, only for $n = \pm 1$ can $z = \pm\sqrt{(2(w - \ln(1+w) - 2n\imath\pi))}$ get somewhat beyond the boundary of $\mathbb{S}$ into a larger domain for the analytic continuation $F(z)$ of $f(z)$. Within this tentatively extended domain $F(z)$ satisfies

$F(z) - \ln(1 + F(z)) \pm 2\imath\pi = z^2/2$ and $F(z) = \exp(F(z) - z^2/2) - 1$ and $F'(z) = z\cdot(1 + 1/F(z))$.

The domain is extended "tentatively" because it contains singularities of $F(z)$ that affect its analytic continuation beyond them. The tentative enlargement lies between the star $\mathbb{S}$ shown above and a bigger star whose boundary is traced by $z = \pm\sqrt{(2(w - \ln(1+w) \pm 2\imath\pi))}$ as w runs along the slit whereon $w < -1$. Here is a picture with singularities marked by "x":



$F(z)$ has singularities inside the tentatively extended domain: $F'(z) = \infty$ wherever $F(z) = 0 \neq z$. The singularities nearest $z = 0$ are *Branch Points* at $\pm z = 2\sqrt{\pm\pi\cdot\imath} = (1\pm\imath)\cdot\sqrt{2\pi} \approx 2.5066\cdot(1\pm\imath)$. These determine the radius of convergence of the Taylor series; it is $r = 2\sqrt{\pi}$. As z traces a path starting at $z = 0$, $F(z)$'s value depends on how the path wends among the singularities.

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This problem illustrates an obstacle impeding the automation of the algebra of transcendental complex analytic functions. Computerized algebra systems like MAPLE, MATHEMATICA and DERIVE can cope with only the simplest domains, if any. For instance, these programs can compute as many as you like of the coefficients of the Taylor series of $F(z)$, but not r . The accurate numerical computation of $F(z)$ is another interesting story for some other day.