0. Exhibit a function $f(z)$, *continuous* (but not analytic) throughout the closed unit disk, to which function $f(z)$ no polynomial can approximate arbitrarily closely thereon.

Answer 0: Try $f(z) := 1 - |z|^2$. It is continuous. But no polynomial P can satisfy $|f(z) - P(z)| < ß < 1/4$ throughout the unit disk. This is so because P would have to satisfy $ß > |f(z) - P(z)| = |P(z)|$ on the disk's boundary $|z| = 1$, and then the Maximum Modulus Theorem would imply $ß > |P(0)|$ too, forcing $|f(0) - P(0)| = |1 - P(0)| > 1 - ß > ß$.

Polynomials can easily approximate arbitrarily closely to any function $f(z)$ analytic throughout the *closed* unit disk. Then the Maclaurin series for f converges in a larger disk of radius bigger than some $r > 1$, and converges in the closed unit disk at least as fast as a geometric series with common ratio $1/r < 1$; this means that the series' first N terms constitute a polynomial that approximates f as closely as desired if N is chosen big enough.

The situation is more interesting when f is analytic in the *open* unit disk but merely continuous towards and along its boundary, or when f is analytic throughout a closed bounded simply connected region with a piecewise smooth boundary. But those are stories for another day about best polynomial approximation.

1(a). As z traverses the unit circle O once counterclockwise, how many times does $z^2 - 1/4$ go around the origin? What is $\int_O 2z\cdot dz/(z^2 - 1/4)$? **Answers**: Twice. $4\pi\imath$.

1(b). As Φ increases from 0 to 2π, the expression $(e^{3\imath\Phi} - 1/2)(2e^{-2\imath\Phi} + 1)$ traces some closed curve in the complex plane. How many times does that curve go around the origin? **Answer**: Once because $(z^3 - 1/2)(2/z^2 + 1)$ has 3 zeros and 2 poles inside the unit circle.

2. Evaluate $\int_E z^7\cdot e^{\sin(z)}\cdot dz$ integrated around the ellipse E traced by $z = 5\cdot\cos(t) + 2\imath\cdot\sin(t)$ as t increases from 0 to 2π. **Answer**: 0, because the integrand is analytic inside and on E.

3. Evaluate $\int_O z\cdot e^{z\cdot\sin(\pi z)}dz/(z - 1/2)$ integrated once counterclockwise around unit circle O.
Answer: $\pi\cdot\imath\cdot\sqrt{e} = 2\pi\cdot\imath\cdot\{$ Residue of $z\cdot e^{z\cdot\sin(\pi z)}/(z - 1/2)$ at $z = 1/2 \}$.

4. Use residues to evaluate real integrals $\int_{-\infty}^{+\infty} dx/(1 + x^2)$ and $\int_{-\infty}^{+\infty} dx/(1 + x^2)^2$, showing what contour you use and which singularities you take into account. **Answers**: π and $\pi/2$. The contour is an infinite semicircle above its diameter, the real axis; the singularity is a pole at $z = \imath$; the residue there is $-\imath/2$ for the first and $-\imath/4$ for the second integral because

$$2/(z^2 + 1) = \imath/(z+\imath) - \imath/(z-\imath) \quad \text{and} \quad 4/(z^2 + 1)^2 = \imath/(z+\imath) - \imath/(z-\imath) - 1/(z+\imath)^2 - 1/(z-\imath)^2 .$$

The next six problems are intended to be solved in sequence.

5(a). Check that cot(z) = cos(z)/sin(z) has a simple pole at $z = n\pi$ for every integer n , and prove that cot(z) has no other singularities in the finite complex z-plane. What is the residue of cot(z) at each of its poles? **Answer**: When $1/\cot(z) = \tan(z) = 0$ its derivative $1 + \tan^2(z) = 1 \ne 0$ so the zeros of tan are all simple poles of cot . Those zeros $z = n\pi$ are the same as the zeros of $2\imath\cdot e^{\imath z}\sin(z) = e^{2\imath z} - 1$ all of which must be real because otherwise $|e^{2\imath z}| \ne 1$. The residue of cot at each pole is 1 because it is the value of cos(z)/sin(z)' there.

5(b). What is the residue of cot(z)/z at each of its poles? Hint: cot is an odd function. **Answer**: cot(z)/z is even, so its residue is 0 at z = 0 ; at $z = n\pi \ne 0$ the residue is $1/(n\pi)$.

6(a). For every integer $n \ge 0$ let C_n be the boundary, traversed counterclockwise, of the z-plane's square in which neither |Re(z)| nor |Im(z)| exceeds $n\pi + \pi/2$. Calculate $\int_{C_n} \cot(z)\cdot dz$. **Answer**: $\int_{C_n} \cot(z)\cdot dz = 2\pi\imath(2n+1) = 2\pi\imath\sum$ (Residues of cot(z) inside C_n) .

6(b). Calculate $\int_{C_n} \cot(z)\cdot dz/z$. **Answer**: $\int_{C_n} \cot(z)\cdot dz/z = 0$ because pairs of residues cancel.

7. Find a constant upper bound M for |cot(z)| on all of C_n by first treating each of its sides separately. Of course, there are many choices available for M ; the smaller the better. Use it to show for any fixed complex w that $\int_{C_n} \cot(z)dz/((z-w)z) \to 0$ as $n \to +\infty$. **Answer**: On each C_n an upper bound $M_n = 1 + 2/(e^{(2n+1)\pi} - 1) < M := 2$. Then for all $n > |w|/\pi$ we find $|\int_{C_n} \cot(z)dz/((z-w)z)| < M\cdot(C_n\text{'s Perimeter})/((n\pi+\pi/2 - |w|)(n\pi+\pi/2)) = 8M/(n\pi+\pi/2 - |w|) \to 0.$

8. What is the residue of cot(z)/(z – w) at each of its poles? Recall problem 5 to see how the answer depends upon whether w is a pole of cot . Suppose now that w lies inside C_n ; what is $\int_{C_n} \cot(z)dz/(z-w)$ and why is it not $2\pi\imath\cdot\cot(w)$? **Answer**: If $w \ne n\pi$ the residue of cot(z)/(z–w) at $z = n\pi$ is $1/(n\pi - w)$, and the residue at z = w is cot(w) ; but if $w = n\pi = z$ the residue at z is 0 . If w lies inside C_n , Cauchy's Integral Formula says that $2\pi\imath\cdot f(w) = \int_{C_n} f(z)dz/(z-w)$ when f is analytic on and inside C_n , as cot is not. Instead, provided w/π is not an integer, $\int_{C_n} \cot(z)dz/(z-w) = 2\pi\imath\cdot(\ \cot(w) + \sum_{-n\le k\le n} 1/(k\pi - w)\ )$. This is an analytic function of w with removable singularities at $w = m\pi$ for integers m between ±n inclusive, for which $\int_{C_n} \cot(z)dz/(z-m\pi) = 2\imath\cdot\sum_{-n\le k\le n\ \&\ k\ne m} 1/(k - m)$.

9. After checking $1/(w-z) = 1/w + z/((w-z)w)$, use this and integrals in previous problems, perhaps with z and w exchanged, to show that $\cot(z) = 1/z + 2\cdot\sum_{k\ge1} z/(z^2 - k^2\pi^2)$. Show that this series for cot(z) converges *uniformly* in any compact region that contains no pole of cot . What does this imply for term-by-term integration of the series? (It's no power series.)

Answer: $z\cdot\int_{C_n} \cot(w)dw/\big((w-z)w\big) = \int_{C_n} \cot(w)\big(1/(w-z) - 1/w\big)\ dw$ … now fix z inside C_n

$= \int_{C_n} \cot(w)dw/(w-z)$ by Problem 6b …

$= 2\pi\imath\cdot(\ \cot(z) + \sum_{-n\le k\le n} 1/(k\pi - z)\ )$ by Problem 8
$= 2\pi\imath\cdot(\ \cot(z) - 1/z + \sum_{1\le k\le n} (1/(k\pi-z) - 1/(k\pi+z))\ )$
$= 2\pi\imath\cdot(\ \cot(z) - 1/z - 2\sum_{1\le k\le n} z/(z^2 - k^2\pi^2)\ )$
$\to 0$ as $n \to +\infty$ by Problem 7.

Therefore $\cot(z) = 1/z + 2\cdot\sum_{k\ge1} z/(z^2 - k^2\pi^2)$. Now restrain z to any compact region R that contains no pole of cot . R must be bounded; say $Z > |z|$ for all z in R . Then for every integer $K > Z/\pi$ we find $|\cot(z) - 1/z - 2\cdot\sum_{1\le k\le K} z/(z^2 - k^2\pi^2)| < 2\cdot\sum_{k>K} Z/(k^2\pi^2 - Z^2)$, which is a convergent infinite series that can be made arbitrarily small by choosing K big enough independently of z in R . Therefore the series for cot(z) converges uniformly in R .

The student should be able to prove, or find printed proofs for, the following theorems:

- A function continuous on a compact region is uniformly continuous thereon.
- A sequence of continuous functions, convergent uniformly on a compact region, converges to a function continuous thereon.
- Integrals, of a sequence of continuous functions convergent uniformly on a compact region to a continuous function, converge to its integral.

Consequently $\int \cot(z)dz = \int dz/z + \sum_{k\ge1} \int 2z\cdot dz/(z^2 - k^2\pi^2)$ along paths of integration in R .

10. Integrate the series for $\cot(z) - 1/z$ to prove that $\sin(z)/z = \prod_{k\ge1} (1 - z^2/(k\pi)^2)$. (This is an instance of the representation of an analytic function by a convergent infinite product instead of infinite sum.) **Answer**: $\cot(z) - 1/z$ has a removable singularity at $z = 0$ where its Maclaurin series begins with $-z/3 - z^3/45 - \dots$. (Can you see why?) Therefore, along any path of integration that avoids points w that are nonzero integer multiples of π , we find that $\int_0^z (\cot(w) - 1/w)dw = \sum_{k\ge1} \int_0^z 2w\cdot dw/(w^2 - k^2\pi^2)$. We may well be tempted to rewrite this as $\ln(\sin(z)/z) = \sum_{k\ge1} \ln(1 - z^2/(k\pi)^2)$, but doing so begs some questions about the branches of the logarithm functions that appear there. To avoid these questions let's keep $|z|$ so tiny that no logarithm's argument can become negative real; this is feasible since $\sin(z)/z = 1 - z^2/6 + \dots$. Then all logarithms can be *Principal* logarithms, and then $\ln(\sin(z)/z)$ is equal to that series of logarithms for all $|z|$ tiny enough, and then taking exp(…) of both sides yields

$\sin(z)/z = \prod_{k\ge1} (1 - z^2/(k\pi)^2)$ for all $|z|$ small enough that no factor is negative real.

This is the equation whose validity we wish now to extend by a kind of analytic continuation to the entire (finite) z-plane. The extension would be immediate if we knew the infinite product to be an entire function, but we don't know that yet. We'll know it soon.

Now recall Problem 9's compact region R , which lies in a big disk $|z| < Z$ for some huge $Z > 0$, and recall integer $K > Z/\pi$. As we saw in Problem 9 , now $\sum_{k>K} 2z/(z^2 - k^2\pi^2)$ converges uniformly throughout that big disk, and so does its integral

$\int_0^z \sum_{k>K} 2w\cdot dw/(w^2 - k^2\pi^2) = \sum_{k>K} \int_0^z 2w\cdot dw/(w^2 - k^2\pi^2) = \sum_{k>K} \ln(1 - z^2/(k\pi)^2)$,

in which now all logarithms are Principal. This integral is an analytic function throughout the big disk because it is path-independent therein; consequently its exp(…) is analytic too. Thus $\prod_{k>K} (1 - z^2/(k\pi)^2)$ is analytic throughout the big disk, and therefore so is $\prod_{k\ge1} (1 - z^2/(k\pi)^2)$. Since the big disk can be arbitrarily big, the infinite product is an entire function, and therefore it equals $\sin(z)/z$ for all finite z , as claimed.