On paper to be supplied, provide answers for these problems without using any computer, text, notes nor communicating device. Number pages you submit in the order you want them read; put the problem's number as well as your name and student I.D. # on each page submitted.

1: Supply a harmonic conjugate v(x, y) for $u(x, y) := (x^2 + y^2 – x)/(x^2 + y^2 – 2x + 1)$ valid over the largest possible domain, and explain how v(x, y) and its domain were determined.

Solution 1: The desired conjugate is $v(x, y) := (\text{Any Constant}) – y/(x^2 + y^2 – 2x + 1)$ over all of the (x, y)-plane except the point (x, y) = (1, 0) . Here is why: Only that point is excluded from the domain of u(x, y) , which is an *Expression* analytic in both x and y regarded as complex variables. And since u(x, y) is harmonic, there is some analytic function $f(z)$ of the complex variable $z = x + ıy$ satisfying $2u(x, y) = f(x + ıy) + \overline{f}(x + ıy)$. Now pretend that z and $\overline{z} = w$ are independent complex variables and solve the two equations $x + ıy = z$ and $x – ıy = \overline{z} = w$ for $x = (z + \overline{z})/2 = (z + w)/2$ and $y = ı·(\overline{z} – z)/2 = ı·(w – z)$, which may now be construed temporarily as complex variables independent of each other. Consequently $2u(x, y) = 2u((z+w)/2, ı·(w–z)/2) = f(z) + \overline{f}(w)$ wherein $\overline{f}(w)$ is an analytic function of w (over a domain that is the complex conjugate of the domain of $f(z)$) by virtue of the Cauchy-Riemann equations that f satisfies. Choose arbitrarily any constant value for $\overline{w}$ in the domain of f to determine a constant (but not yet known) $\overline{f}(\overline{w})$ and obtain a slightly circular formula $f(z) := 2u((z+w)/2, ı·(w–z)/2) – \overline{f}(\overline{w})$ for an analytic *Expression* $f(z) = f(x + ıy)$ of which harmonic u(x, y) is the real part when x and y are real. This formula leaves, as it should, $\text{Im}(\overline{f}(\overline{w}))$ undetermined but does determine $\text{Re}(\overline{f}(\overline{w})) = (\overline{f}(\overline{w}) + f(\overline{w}))/2 = u(\text{Re}(\overline{w}), \text{Im}(\overline{w}))$, so

$$f(z) = 2u((z+w)/2, ı·(w–z)/2) – u(\text{Re}(\overline{w}), \text{Im}(\overline{w})) + (\text{Arbitrary Imaginary Constant}) .$$

For u(x, y) given above this yields $f(z) = z/(z–1) + (\text{An Imaginary Constant})$ *provided* $w \neq 1$, whence $v(x, y) = \text{Im}(f(x + ıy))$ *etc.* as claimed, and without having to integrate derivatives.

Cf. Review Ex. #9, §2.R, and Ex. 32, §1.5 of Basic Complex Analysis 3rd ed. by J.E. Marsden & M.J. Hoffman (1999). See also Ex. 19, p. 14 of <www.cs.berkeley.edu/~wkahan/Math185/Derivative.pdf>, and pp. 264-7 of Z. Nehari's text *Intro. to Complex Analysis* Rev'd Ed. (1969) for a proof of the foregoing method's general validity.

An alternative solution obtains v(x, y) by integrating Grad(v(x, y)) . The Cauchy-Riemann equations provide $\text{Grad}(v(x, y)) := [v_x, v_y] = [–u_y, u_x]$ wherein the subscripts denote partial derivatives like $v_x(x, y) := \partial v(x, y)/\partial x$. The variable $\zeta := \xi + ı\eta$ of integration runs from an arbitrary point $c := a + ıb$ in v's domain to $z := x + ıy$ along a dog-leg path in the (x, y)-plane from (a, b) to (x, b) to (x, y) . Along this path $v(\xi, \eta)$ changes by an amount

$$v(x, y) – v(a, b) = \int_c^z \text{Grad}(v(\xi, \eta))\bullet[d\xi, d\eta] = –\int_a^x u_y(\xi, b)·d\xi + \int_b^y u_x(x, \eta)·d\eta .$$

Given the problem's $u(x, y) := 1 + (x–1)/\big((x–1)^2 + y^2\big)$, we find after some algebraic effort that

$$\int u_x(x, \eta)·d\eta = \int (\eta^2 – (x–1)^2)·d\eta/(\eta^2 + (x–1)^2)^2 = –\eta/\big(\eta^2 + (x–1)^2\big) \quad \text{and}$$

$$\int u_y(\xi, b)·d\xi = –2b·\int (\xi–1)·d\xi/(b^2 + (\xi–1)^2)^2 = b/\big(b^2 + (\xi–1)^2\big) ;$$

and then the expression for v(x, y) simplifies, after little more algebraic effort, to

$$v(x, y) = v(a, b) + b/\big(b^2 + (a-1)^2\big) – y/\big(y^2 + (x–1)^2\big) \quad \text{as claimed.}$$

Integrating along a different path from (a, b) to (a, y) to (x, y) produces the same result even if the two paths are separated by the sole singularity of v(x, y) at (x, y) = (1, 0) .

2: $f(z)$ is analytic over some open domain including a point z_o at which $f'(z_o) \ne 0$. Explain why $2\pi\imath/f'(z_o) = \int dz/(f(z) - f(z_o))$ integrated around any sufficiently small circle centered at z_o . You may invoke the *Inverse Function Theorem* and *Integration-by-Parts* if you wish.

Solution 2: Let $f(z)$ and $g(w)$ be two analytic functions inverse to each other in some open neighborhoods of $z \approx z_o$ and $w \approx w_o := f(z_o)$ wherein $z \equiv g(f(z))$ and $w \equiv f(g(w))$; therein $1 \equiv g'(f(z))\cdot f'(z) \equiv f'(g(w))\cdot g'(w)$ by virtue of the *Inverse Function Theorem*.

See pp. 7-9 on the class web page's <…/Derivative.pdf>, or pp. 69-71 and 400-401 of *Basic Complex Analysis* 3rd ed. by Marsden & Hoffman, and their Ex. #10, §2.R .

Let γ be a circle centered at z_o and so small that its image $f(\gamma) \approx w_o + f'(z_o)\cdot(\gamma - z_o)$ is a tiny closed curve enclosing w_o . Then substitute $w = f(z)$ into the Cauchy Integral Formula's …

$$2\pi\imath/f'(z_o) = 2\pi\imath\cdot g'(w_o) = \int_{f(\gamma)} g'(w)\cdot dw/(w - w_o) = \int_\gamma dz/(f(z) - f(z_o)) \text{ as claimed.}$$

An alternative solution differentiates Cauchy's $2\pi\imath\cdot g(w_o) = \int_{f(\gamma)} g(w)\cdot dw/(w - w_o)$ to obtain $2\pi\imath\cdot g'(w_o) = \int_{f(\gamma)} g(w)\cdot dw/(w - w_o)^2$ and then substitutes $w := f(z)$ to deduce that $2\pi\imath/f'(z_o) = \int_\gamma z\cdot f'(z)\cdot dz/(f(z) - f(z_o))^2 = -\int_\gamma z\cdot d\big(1/(f(z) - f(z_o))\big)$, after which *Integration-by-Parts* yields $2\pi\imath/f'(z_o) = -\Delta_\gamma[\, z/(f(z) - f(z_o))\,] + \int_\gamma dz/(f(z) - f(z_o)) = \int_\gamma dz/(f(z) - f(z_o))$, again as claimed.

3: Show how to evaluate $\int_{-\pi}^{\pi} \exp(-\imath\theta)\cdot\exp(\exp(\imath\theta))\cdot d\theta$.

Solution 3: Call the integral $J := \int_{-\pi}^{\pi} \exp(-\imath\theta)\cdot\exp(\exp(\imath\theta))\cdot d\theta$. Hereunder is why $J = 2\pi$:
Let O be the z-plane's unit circle on which $z = \exp(\imath\theta)$, and let w lie strictly inside O .
The Cauchy Integral Formula provides $2\pi\imath\cdot\exp(w) = \int_O \exp(z)\cdot dz/(z - w)$ and its derivative $2\pi\imath\cdot\exp(w) = 2\pi\imath\cdot d\exp(w)/dw = \int_O \exp(z)\cdot dz/(z - w)^2$ which the substitution $z := \exp(\imath\theta)$ turns into $2\pi\imath\cdot\exp(w) = \int_{-\pi}^{\pi} \exp(\exp(\imath\theta))\cdot\imath\cdot\exp(\imath\theta)\cdot d\theta/(\exp(\imath\theta) - w)^2$. Set $w := 0$ to find $2\pi\imath = \imath\cdot J$.

This problem is Ex. #11, §2.R, of the text *Basic Complex Analysis* 3rd ed. by Marsden & Hoffman.