Solutions for the Midterm Test

On paper to be supplied, answer any four of these problems without using any computer, text, notes nor communicating device. Number pages you submit in the order you want them read; put the problem's number as well as your name and student I.D. # on each page submitted.

1: Möbius map $M(z) := (\mu z + ß)/(b - mz)$ with $ç := ß\cdot m + b\mu \neq 0$ has real coefficients μ, $ß$, b and m. How does the sign of $ç$ determine where M maps the upper half-plane? Why?

Solution 1: M maps the closed (at ∞) real axis to itself in the complex plane $\mathbb{C}$, and maps its upper half-plane to itself if $ç > 0$, or to its lower half-plane if $ç < 0$. Here is why: Rewrite $M(z) = (ç/m)/(b - m\cdot z) - \mu/m$ to derive $M'(z) = ç/(b - m\cdot z)^2$, which has the same sign as $ç$ has. Except from the pole $z = b/m$, if z moves from a point on the real axis into the upper half-plane the image $M(z)$ does likewise if $ç > 0$, or oppositely into the lower half-plane if $ç < 0$. And then, because M is a bijection of the Riemann Sphere onto itself, $M(z)$ cannot pass from one half-plane's interior to the other's without z crossing the real axis again.

Alternative Solution 1: $2\imath\cdot\mathrm{Im}(M(z)) = M(z) - \overline{M}(z) = M(z) - M(\bar{z})$. Set $z := x + \imath y$ to get $\mathrm{Im}(M(x + \imath y)) = ç\cdot y/|b - m\cdot(x + \imath y)|^2$, whence $\mathrm{sign}(\mathrm{Im}(M(z))) = \mathrm{sign}(ç\cdot\mathrm{Im}(z))$. *Etc.*

2: Suppose $g(x, y) = g(x, -y)$ is harmonic on a domain whose interior includes a segment of the real (x-) axis. Why must some harmonic conjugate h of g satisfy $h(x, y) = -h(x, -y)$?

Solution 2: $g(x, y)$ is harmonic inside a domain $\mathbb{D} = \overline{\mathbb{D}}$ which is its own reflection in the real axis because $g(x, -y) = g(x, y)$. A function $f(z)$ analytic inside $\mathbb{D}$ has $\mathrm{Re}(f(x + \imath y)) = g(x, y)$ and $\mathrm{Im}(f(x + \imath y)) = h(x, y) + (\text{arbitrary constant})$. Now, $\overline{f}(\bar{z})$ is another analytic function of z inside $\mathbb{D}$, and so is $c(z) := f(z) - \overline{f}(\bar{z})$. Therein $\mathrm{Re}(c(x + \imath y)) = g(x, y) - g(x, -y) = 0$ has a Constant as its harmonic conjugate. Therefore $\mathrm{Im}(c(x + \imath y)) = h(x, y) + h(x, -y) + (\text{constant})$ must be that Constant, which is arbitrary. Therefore $h(x, y) + h(x, -y) := 0$ is as good a choice of constant as any other. This answers the question.

3: Suppose $f(z)$ is analytic throughout the finite complex plain but $|f(z)| \to \infty$ as $|z| \to \infty$. Must $f(z)$ be a polynomial function of z ? Why?

Solution 3: Yes. The Taylor series of $f(z)$ expanded about $z = 0$ converges for all finite z and provides coefficients for a Laurent expansion about $w = 0$ of $q(w) := f(1/w)$ which must converge for all finite w except $w = 0$ where $q(w)$ has an isolated singularity. It cannot be an essential singularity because $q(w) \to \infty$ as $w \to 0$, contrary to how the Casorati-Weierstrass theorem says q would behave if its singularity were essential. Therefore $q(w)$ has a pole of finite order at $w = 0$, and this order matches the degree of what we infer to be a polynomial f.

4: Explain what is wrong with the following reasoning: " Since $\mu^z := \exp(z\cdot\log(\mu))$, so $d(\mu^z)/dz = (\log(\mu))\cdot\mu^z$. Also $d(\mu^z)/dz = z\cdot\mu^{z-1}$. Therefore $z\cdot\mu^{z-1} = (\log(\mu))\cdot\mu^z$, whence follows $z = \mu\cdot\log(\mu)$."

Solution 4: $\partial(\mu^z)/\partial z \neq z\cdot\mu^{z-1} = \partial(\mu^z)/\partial\mu$. If you overlooked this you should consider some other way to earn your living. (This is #16 on p. 314 of the text by Marsden & Hoffman.)

5: $f(z)$ is the analytic root of the equation $f - \log(1 + f) = z^2/2$ for the principal branch of log with $f'(0) = 1$. What is the radius of convergence of $f(z)$'s Taylor series expansion around $z = 0$, and why?

Solution 5: The radius of convergence is $2\sqrt{\pi}$; here is why: $f = \infty$ only at $z = \infty$, so $f(z)$ has no pole in the finite z-plane. On the other hand, f and its analytic continuations have isolated singularities at *Branch-points* where a derivative is infinite. Implicit differentiation provides $f'(z) = (1 + f(z))\cdot z/f(z)$ which is finite unless $z \neq f(z) = 0$. This happens to a branch $f_n(z)$ satisfying $f_n - \log(1 + f_n) + 2n\pi\imath = z^2/2$ only where $z = \pm 2\sqrt{n\pi\imath}$ for $\pm n = 1, 2, 3, \dots$. The value $n = 0$ is omitted because $z = 0$ is not a singularity of $f(z) = f_0(z)$ since $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2/3$, ... are all obtained by differentiating the given equation repeatedly. There are four nearest branch-points $z = \pm 2\sqrt{\pm\pi\imath}$ all at a distance $2\sqrt{\pi}$ from the center $z = 0$ of the Taylor series. The desired radius is that least distance. For more details and pictures see `<www.cs.berkeley.edu/~wkahan/Math185/Solvef.pdf>` on the class web page.

6: Suppose $f(z)$ is analytic with a simple pole at $z = 0$ and residue μ there. For $r > 0$ and $\Theta > 0$ let $C(r,\Theta)$ be the arc traced by $z = r\cdot e^{\imath\theta}$ as θ increases from 0 to Θ . Evaluate

$$\lim_{r\to 0} \int_{C(r,\Theta)} f(z)\cdot dz .$$

Solution 6: The limit is $\mu\cdot\Theta\cdot\imath$. Here is why: The Laurent expansion around $z = 0$ of $f(z)$ must have the form $f(z) = \mu/z + \sum_{n\geq 0} a_n\cdot z^n$ so, if r is smaller than the series' radius of convergence, $J(r) := \int_{C(r,\Theta)} f(z)\cdot dz = \mu\cdot\int_{C(r,\Theta)} dz/z + \sum_{n\geq 0} a_n\cdot\int_{C(r,\Theta)} z^n\cdot dz$. For $n \geq 0$

$$\int_{C(r,\Theta)} z^n\cdot dz = \{\text{Change in } z^{n+1}/(n+1) \text{ along } C(r,\Theta)\} = r^{n+1}\cdot(e^{n\imath\Theta} - 1) \to 0 \text{ as } r \to 0+ .$$

For $n = -1$

$$\int_{C(r,\Theta)} dz/z = \{\text{Change in } \log(z) \text{ along } C(r,\Theta)\}$$
$$= \{\text{Change in } \log(r) + \imath\theta \text{ as } \theta \text{ runs from } 0 \text{ to } \Theta\} = \imath\Theta .$$

Consequently $J(r) \to \mu\cdot\Theta\cdot\imath$ as $r \to 0$.

Compare Lemma 4.3.10 on p.285 in *Basic Complex Analysis* by J. Marsden & M.J. Hoffman.