1: What are the Cauchy-Riemann Equations and why do they matter in this course? (5 min.)

Answer 1: Let \( f(z) = f(x + iy) = g(x, y) + ih(x, y) \) exhibit the partition of a complex function \( f \) and its complex argument \( z = x + iy \) into their respective real and imaginary parts. Then the Cauchy-Riemann equations are \( \frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \) and \( \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x} \). Within any open region in the plane whereon these are satisfied, \( f \) has a complex derivative \( f' = df/dz = \partial g/\partial x + i\partial h/\partial x \) in the sense that, for each \( z = x + iy \) in the region, \( (f(z+w) - f(z) - f'(z)w)/w \to 0 \) as \( w \to 0 \) no matter what path in the complex plane \( w \) follows as it approaches 0 (Our text lets \( \Phi(z) = f(z) \)).

The same results can be obtained by more laborious methods. If \( \Phi(x + iy) := \Phi(x, y) := g(x, –y) – ih(x, y) \), then \( \Phi(\bar{w} + z/2) = \Phi(w – z/2) \) for any complex constant \( w \) and variable \( z \) produces \( f(z) = \frac{1}{2} \left( \Phi(w + z) + \Phi(w + z) + \Phi(w + z) - \Phi(w + z) \right) \). From our \( g \) comes \( f(z) = \sqrt{2} \left( 2\sqrt{w + z} + 2\sqrt{(w + z)^2 - (w - z)^2} - \Phi(w) \right) \). Principal Values for the square roots introduce slits, one along the ray \( z/w < 0 \), into the domain of \( f(z) \). To match its domain to \( g \)'s we must choose \( w \geq 0 \) getting \( f(z) = \frac{1}{\sqrt{2}} \sqrt{2w + 2z} \). This simplifies, because \( w \geq 0 \) and not \( z \leq 0 \), to \( f(z) = \sqrt{2w} + \sqrt{2z} - \Phi(w) \). Because we

2: Upon what domain is \( g(z) := \sqrt{(x + \sqrt{(x^2 + y^2})} \) Harmonic, and why, and what is one of its Harmonic Conjugates on that domain? (15 min. if done artfully; otherwise 25 min.)

Answer 2: The domain on which \( g(x, y) := \sqrt{(x + \sqrt{(x^2 + y^2})} \) is harmonic is the whole \( (x, y) \)-plane except the semi-axis where \( x \leq 0 \) because neither square root’s argument vanishes in this slitted domain, so all derivatives are finite there. The non-vanishing is evident when \( x > 0 \) and \( -\pi < \theta < \pi \) inside this slitted domain where \( g(r\cdot\cos(\theta), r\cdot\sin(\theta)) = \sqrt{2r}\cdot\cos(\theta/2) = \Re\{\sqrt{2z}\} \), so \( \Im\{\sqrt{2z}\} = \sqrt{2r}\cdot\sin(\theta/2) = \Re\{\sqrt{2z}\} \).

To check the correctness of this answer by testing whether \( g \) and \( h \) satisfy the Cauchy-Riemann equations, note first that \( g\cdot h = y \) and \( g^2 - h^2 = 2x \), so \( g^2 + h^2 = 2\sqrt{(x^2 + y^2)} > 0 \). Differentiating the first two equations leads to a linear system of equations for \( g_x := \partial g/\partial x \), \( g_y := \partial g/\partial y \), etc.:

\[
\begin{bmatrix}
g_x & g_y \\
-\bar{g} & -\bar{h}
\end{bmatrix}
\begin{bmatrix}
g_x & g_y \\
h_x & h_y
\end{bmatrix}
= \begin{bmatrix}1 & 0 \0 & 1\end{bmatrix}, \quad \text{whence} \quad \begin{bmatrix}
g_x & g_y \\
h_x & h_y
\end{bmatrix}
= \begin{bmatrix}
g_x & h_x \\
h_y & -h_x
\end{bmatrix}/(h^2 + g^2), \quad \text{and then} \quad g_x = h_y \quad \text{and} \quad g_y = -h_x.
\]

The same results can be obtained by more laborious methods. If \( f(x + iy) := g(x, y) + ih(x, y) \) is holomorphic in that part of the complex \( z \)-plane corresponding to \( z = x + iy \) where \( g(x, \pm y) \) is harmonic, so is \( \Phi(x + iy) := g(x, –y) – ih(x, –y) \). Therefore \( f(x + iy) = 2g(x, y) – \Phi(x – iy) \), after \( h \) is eliminated, and then substituting \( x = (w+z)/2 \) and \( y = (w-z)/2 \) for any complex constant \( w \) and variable \( z \) produces \( f(z) = 2g((w+z)/2, (w-z)/2) – \Phi(w) \) from our \( g \) comes \( f(z) = \sqrt{2(\sqrt{w} + \sqrt{z})} - \Phi(w) \). Principle Values for the square roots introduce slits, one along the ray \( z/w < 0 \), into the domain of \( f(z) \).
defined $F(z) := f(z)$, we find $F(w) = f(w) = 2\sqrt{2}w - F(w)$ whence $\text{Re}\{F(w)\} = \sqrt{2}w$ and $\text{Im}\{F(w)\}$ is an arbitrary real constant, so $f(z) = \sqrt{2}z + i(\text{real constant})$. This yields $h(x, y) := \text{Im}\{f(x + iy)\} = y / g(x, y) + \text{const.}$ to be a harmonic conjugate of $g(x, y)$, as before.

This question was suggested at the end of the class notes “Solution for Ex. 19 in Derivative.pdf” <Ex19.pdf>.

3: Suppose $f$ is holomorphic in an open region including the closed unit disk $|z| \leq 1$ and maps this disk into itself; explain why $|f'(z)| \leq 1/(1 - |z|^2) \leq 1/(1 - |z|)$ inside this disk. (20 min.)

**Answer 3:** Cauchy’s Integral Formula says $f(z) = \frac{1}{2\pi i} \int_C f(w) \cdot dw / (w - z)$, integrated counterclockwise around the unit circle $C$ that bounds the unit disk, for every $z$ inside this disk. This equation’s derivative yields $f'(z) = \frac{1}{2\pi} \int_{\pi}^{\pi} f(w) \cdot dw / (w - z)^2$. Now, $w = e^{i\theta}$ for $-\pi \leq \theta \leq \pi$ and $|f(w)| \leq 1$ as $w$ runs around $C$. Consequently $|f'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 / (1 - 2r \cdot \cos(\theta - \pi) + r^2$).

Next write $z = r \cdot e^{i\psi}$ for $0 \leq r := |z| < 1$ and some real $\psi$, so $|e^{i\theta} - z|^2 = 1 - 2r \cdot \cos(\theta - \psi) + r^2$. Set $\theta := \phi - \psi$ to get $|f'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 / (1 - 2r \cdot \cos(\theta) + r^2)$. This trigonometric integral turns into a rational integral after the customary substitution $t := \tan(\theta/2)$, whence $d\theta = 2dt / (1 + t^2)$ and $\cos(\theta) = (1 - t^2) / (1 + t^2)$ and then $|f'(z)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} 2dt / ((1-r)^2 + (1+r)^2 \cdot t^2) = 1/(1 - r^2)$, which is the inequality requested.

This question’s answer is sharper and easier in some respects than our text’s Ex. 2 on p. 82, but generally weaker than the inequality $|f'(z)| \leq (1 - |f(z)|^2) / (1 - |z|^2)$ in Ex. 3 on p. 88 whose proof also answers this question.

4: Suppose $f$ is an invertible, continuously once differentiable, and **Conformal** (preserves the direction and magnitude of angles) map of the **Riemann Sphere** onto itself; why must $f(z)$ be a **Möbius** (linear-fractional) function of $z$ in the complex plane closed by a point at $\infty$? (15 min.)

**Solution 4:** Let $g(z) := M(f(z))$ wherein $M$ is a Möbius function that maps $f(0)$, $f(1)$ and $f(\infty)$ (all distinct because $f$ is invertible) onto $0$, $1$ and $\infty$ respectively. This $g(z)$ remains finite throughout the complex $z$-plane because $g(z) = \infty$ only where $z = f^{-1}(M^{-1}(\infty)) = \infty$. Also $g$ has continuous partial derivatives inherited from those of $f(z)$ or $f(1/z)$ or $1/f(z)$ or $1/f(1/z)$ regarded as functions from the sphere to itself. And $g$ maps the plane conformally onto itself, so (by the text’s “CONFORMAL IMPLIES HOLOMORPHIC” theorem on p. 21) $g(z)$ must be holomorphic everywhere in the complex $z$-plane. The same goes for $G(z) := 1 / g(1/z)$. Now set $h(z) := g(z)/z$ except that singularities are removed by setting $h(0) := g'(0)$ and $h(\infty) := 1/G'(0)$ which is finite because $G(z)$ can’t be conformal at $z = 0$ if $G'(0) = 0$. This continuous $h(z)$ is an entire function bounded by its maximum magnitude over the compact sphere, so Liouville’s theorem implies that $h(z)$ is constant. The constant is $h(1) = 1$, so $f(z) = M^{-1}(z)$ as claimed.

I think the assumption that $f$ is differentiable on the sphere is superfluous; continuity is actually sufficient.