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Midterm Exam for Math. 185 §2

1:15 - 2 pm, 30 Oct. 2006

Bring no books, notes, papers, computers nor communicating instruments to this exam. Answer as many of its questions as you can in the time allotted. Write each answer *Legibly* on its own page(s) of the blanks supplied, and PRINT its number and YOUR NAME at the top. A complete answer is worth more than a few partial answers.

1: What are the *Cauchy-Riemann Equations* and why do they matter in this course? (5 min.)

Answer 1: Let $f(z) = f(x + \imath y) = g(x, y) + \imath h(x, y)$ exhibit the partition of a complex function f and its complex argument $z = x + \imath y$ into their respective real and imaginary parts. Then the Cauchy-Riemann equations are $\partial g/\partial x = \partial h/\partial y$ and $\partial g/\partial y = -\partial h/\partial x$. Within any open region in the plane whereon these are satisfied, f has a complex derivative $f' = df/dz = \partial g/\partial x + \imath \partial h/\partial x$ in the sense that, for each $z = x + \imath y$ in the region, $\big(f(z+w) - f(z) - f'(z)\cdot w\big)/w \to 0$ as $w \to 0$ no matter what path in the complex plane w follows as it approaches 0 . (Our text lets f have a complex derivative at one point if the Cauchy-Riemann equations are satisfied at it.)

2: Upon what domain is $g(x, y) := \sqrt{\big(x + \sqrt{(x^2 + y^2)}\big)}$ *Harmonic*, and why, and what is one of its *Harmonic Conjugates* on that domain? (15 min. if done artfully; otherwise 25 min.)

Answer 2: The domain on which $g(x, y) := \sqrt{\big(x + \sqrt{(x^2 + y^2)}\big)}$ is harmonic is the whole (x, y)-plane except the semi-axis where $x \le 0$ because neither square root's argument vanishes in this slitted domain, so all derivatives are finite there. The non-vanishing is evident when $x > 0$; otherwise rewriting $g(x, y) = |y|/\sqrt{\big(-x + \sqrt{(x^2 + y^2)}\big)}$ for $x \le 0 \ne y$ shows that, though $g(x, y)$ is continuous, $\partial g(x, y)/\partial y$ is discontinuous as y passes through zero. $z = r\cdot\exp(\imath\theta)$ for $r > 0$ and $-\pi < \theta < \pi$ inside this slitted domain where $g(r\cdot\cos(\theta), r\cdot\sin(\theta)) = \sqrt{2r}\cdot\cos(\theta/2) = \mathrm{Re}\{\sqrt{2z}\}$, so $\mathrm{Im}\{\sqrt{2z}\} = \sqrt{2r}\cdot\sin(\theta/2) = h(x, y) := y/g(x, y) = \sqrt{\big(-x + \sqrt{(x^2 + y^2)}\big)}\cdot\mathrm{signum}(y)$ is a harmonic conjugate of g . Other harmonic conjugates of g differ from h by constants.

To check the correctness of this answer by testing whether g and h satisfy the Cauchy-Riemann equations, note first that $g\cdot h = y$ and $g^2 - h^2 = 2x$, so $g^2 + h^2 = 2\sqrt{(x^2 + y^2)} > 0$. Differentiating the first two equations leads to a linear system of equations for $g_x := \partial g/\partial x$, $g_y := \partial g/\partial y$, *etc.*:

$$\begin{bmatrix} h & g \\ g & -h \end{bmatrix} \cdot \begin{bmatrix} g_y & g_x \\ h_y & h_x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{whence} \quad \begin{bmatrix} g_y & g_x \\ h_y & h_x \end{bmatrix} = \begin{bmatrix} h & g \\ g & -h \end{bmatrix} / (h^2 + g^2), \quad \text{and then } g_x = h_y \text{ and } g_y = -h_x .$$

The same results can be obtained by more laborious methods. If $f(x + \imath y) := g(x, y) + \imath h(x, y)$ is holomorphic in that part of the complex z-plane corresponding to $z = x + \imath y$ where $g(x, \pm y)$ is harmonic, so is $\Phi(x + \imath y) := g(x, -y) - \imath h(x, -y)$. Therefore $f(x + \imath y) = 2g(x, y) - \Phi(x - \imath y)$, after h is eliminated, and then substituting $x = (\overline{w}+z)/2$ and $y = \imath(\overline{w}-z)/2$ for any complex constant w and variable z produces $f(z) = 2g((\overline{w}+z)/2, \imath(\overline{w}-z)/2) - \Phi(\overline{w})$. From our g comes $f(z) = \sqrt{\big(2(\overline{w}+z) + 2\sqrt{((\overline{w}+z)^2 - (\overline{w}-z)^2)}\big)} - \Phi(\overline{w}) = \sqrt{\big(2\overline{w} + 2z + 4\sqrt{(\overline{w}\cdot z)}\big)} - \Phi(\overline{w})$. *Principal Values* for the square roots introduce slits, one along the ray $z/w < 0$, into the domain of $f(z)$. To match its domain to g 's we must choose $w \ge 0$ getting $f(z) = \sqrt{\big((\sqrt{2w} + \sqrt{2z})^2 \big)} - \Phi(w)$. This simplifies, because $w \ge 0$ and not $z \le 0$, to $f(z) = \sqrt{2w} + \sqrt{2z} - \Phi(w)$. Because we defined $\Phi(z) := \overline{f}(\bar{z})$, we find $\Phi(w) = \overline{f}(\overline{w}) = \overline{f}(w) = 2\sqrt{2w} - \overline{\Phi}(w)$ whence $\mathrm{Re}\{\Phi(w)\} = \sqrt{2w}$ and $\mathrm{Im}\{\Phi(w)\}$ is an arbitrary real constant, so $f(z) = \sqrt{2z} + \imath\cdot$(real constant). This yields $h(x, y) := \mathrm{Im}\{f(x+\imath y)\} = y/g(x, y) + \text{const.}$ to be a harmonic conjugate of $g(x, y)$, as before.

This question was suggested at the end of the class notes "Solution for Ex. 19 in Derivative.pdf" <Ex19.pdf>.

3: Suppose f is holomorphic in an open region including the closed unit disk $|z| \le 1$ and maps this disk into itself; explain why $|f'(z)| \le 1/(1-|z|^2) \le 1/(1-|z|)$ inside this disk. (20 min.)

Answer 3: Cauchy's Integral Formula says $f(z) = \frac{1}{2\pi\imath}\cdot\int_C f(w)\cdot dw/(w-z)$, integrated counter-clockwise around the unit circle C that bounds the unit disk, for every z inside this disk. This equation's derivative yields $f'(z) = \frac{1}{2\pi\imath}\cdot\int_C f(w)\cdot dw/(w-z)^2$. Now, $w = e^{\imath\phi}$ for $-\pi \le \phi \le \pi$ and $|f(w)| \le 1$ as w runs around C. Consequently $|f'(z)| \le \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} 1\cdot d\phi/|e^{\imath\phi} - z|^2$.

Next write $z = r\cdot e^{\imath\psi}$ for $0 \le r := |z| < 1$ and some real ψ, so $|e^{\imath\phi} - z|^2 = 1 - 2r\cdot\cos(\phi-\psi) + r^2$. Set $\theta := \phi-\psi$ to get $|f'(z)| \le \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} 1\cdot d\theta/(1 - 2r\cdot\cos(\theta) + r^2)$. This trigonometric integral turns into a rational integral after the customary substitution $t := \tan(\theta/2)$, whence $d\theta = 2dt/(1+t^2)$ and $\cos(\theta) = (1-t^2)/(1+t^2)$ and then $|f'(z)| \le \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} 2dt/\big((1-r)^2 + (1+r)^2\cdot t^2\big) = 1/(1-r^2)$, which is the inequality requested.

This question's answer is sharper and easier in some respects than our text's Ex. 2 on p. 82, but generally weaker than the inequality $|f'(z)| \le \big(1-|f(z)|^2\big)/\big(1-|z|^2\big)$ in Ex. 3 on p. 88 whose proof also answers this question.

4: Suppose f is an invertible, continuously once differentiable, and *Conformal* (preserves the direction and magnitude of angles) map of the *Riemann Sphere* onto itself; why must $f(z)$ be a *Möbius* (linear-fractional) function of z in the complex plane closed by a point at ∞ ? (15 min.)

Solution 4: Let $g(z) := M(f(z))$ wherein M is a Möbius function that maps $f(0)$, $f(1)$ and $f(\infty)$ (all distinct because f is invertible) onto 0, 1 and ∞ respectively. This $g(z)$ remains finite throughout the complex z-plane because $g(z) = \infty$ only where $z = f^{-1}(M^{-1}(\infty)) = \infty$. Also g has continuous partial derivatives inherited from those of $f(z)$ or $f(1/z)$ or $1/f(z)$ or $1/f(1/z)$ regarded as functions from the sphere to itself. And g maps the plane conformally onto itself, so (by the text's "CONFORMAL IMPLIES HOLOMORPHIC" theorem on p. 21) $g(z)$ must be holomorphic everywhere in the complex z-plane. The same goes for $G(z) := 1/g(1/z)$. Now set $h(z) := g(z)/z$ except that singularities are removed by setting $h(0) := g'(0)$ and $h(\infty) := 1/G'(0)$ which is finite because $G(z)$ can't be conformal at $z = 0$ if $G'(0) = 0$. This continuous $h(z)$ is an entire function bounded by its maximum magnitude over the compact sphere, so Liouville's theorem implies that $h(z)$ is constant. The constant is $h(1) = 1$, so $f(z) = M^{-1}(z)$ as claimed.

I think the assumption that f is differentiable on the sphere is superfluous; continuity is actually sufficient.