On paper to be supplied, answer as many of these problems as you can with no computer, text, notes nor communicating device. Number pages you submit in the order you want them read; put the problem's number as well as your name and student I.D. # on each page submitted.

Each problem is worth as much as every other.

1: For given complex constants b and $c \neq 0$ provide an expression " $z = \ldots$ " for all the solutions z of the equation " $\text{Real}(c \cdot z - b) = 0$ " and describe the curve(s) they trace.

Solution 1: Every solution $z = (b + \imath\rho)/c$ for a real ρ and, as it runs through all real values, z traces a straight line in the z-plane through b/c perpendicular to $\bar{c}$ (the complex conjugate).

2: Suppose $\partial\mathbb{D}$ is a simple closed curve, the boundary of a finite region $\mathbb{D}$ in the complex plane; and suppose $f_n(z)$ is analytic inside $\mathbb{D}$ and continuous throughout $\mathbb{D}\cup\partial\mathbb{D}$ for every $n = 1, 2, 3, \ldots$. Suppose further that, when w is restricted to $\partial\mathbb{D}$, the infinite sequence $\{ f_n(w) \}$ converges uniformly to a function on $\partial\mathbb{D}$. Must the sequence $\{ f_n(z) \}$ converge to an analytic function inside $\mathbb{D}$? Justify your answer by a proof or a counter-example.

Solution 2: Yes, $\{ f_n(z) \}$ does converge, and does so uniformly, to some analytic function inside $\mathbb{D}$. Here is why: Given any tiny $\delta > 0$, no matter how tiny, there is some big integer $L(\delta)$ such that $|f_m(w) - f_n(w)| < \delta$ at all w in $\partial\mathbb{D}$ for all $m \geq n > L(\delta)$ because convergence is uniform there. The *Maximum Modulus Principle* implies that $|f_m(z) - f_n(z)| < \delta$ at all z in $\mathbb{D}$ for all $m \geq n > L(\delta)$; this is Cauchy's criterion for the (uniform) convergence of the sequence $\{ f_n(z) \}$ to a limit $f(z)$ in in $\mathbb{D}$. And, because convergence is uniform, $f(z)$ is analytic inside $\mathbb{D}$ too. (*Cf.* the Marsden-Hoffman text's problem #20 at the end of §3.2.)

3: Suppose non-constant function $f(z)$ is analytic on some closed bounded domain $\mathbb{D}$. Can $f(z)$ vanish infinitely often on $\mathbb{D}$? Justify your answer by a proof or an example.

Solution 3: No, $f(z)$ cannot vanish infinitely often on $\mathbb{D}$. Here is why: Because $f(z)$ is analytic on the closed domain $\mathbb{D}$ it is also analytic everywhere strictly inside a slightly bigger open bounded domain $\mathbb{B}$ obtained as the union of finitely many (selected from the infinitely many) open disks covering the *Compact* domain $\mathbb{D}$ with $f(z)$ analytic inside every such disk.

Now suppose for the sake of an argument by contradiction that $f(z)$ had infinitely many zeros in $\mathbb{D}$. Because $\mathbb{D}$ is compact, some subsequence of those zeros would converge to a point z_0 in $\mathbb{D}$. And $f(z_0) = 0$ because $f(z)$ is analytic (and thus continuous) on $\mathbb{D}$. From $f(z)$'s Taylor series centered at z_0 we would infer that it is an *Isolated* zero of $f(z)$ in the sense that it is the only zero in some small disk in $\mathbb{B}$ centered at z_0. But then z_0 could not be a limit of $f(z)$'s zeros. Contradiction! (*Cf.* the Marsden-Hoffman text's corollary §3.2.10.)

4: Exhibit any *Möbius* (*Linear Fractional*) map $w = W(z) := (\mu\cdot z + \beta)/(b - \mu\cdot z)$ that takes the disk $|z - 4| \le 3$ to the disk $|w + 1| \le 2$ and the point $z = 1$ to $w = 1$. Which, if any, other Möbius maps fulfil those requirements? Why? You may take for granted that every Möbius map of the complex t-plane's unit circle ($|t| = 1$) onto itself has the form $e^{\imath\theta}\cdot(t - q)/(1 - \bar{q}\cdot t)$ for some real constant θ and some real or complex constant q with $|q| \ne 1$. And don't waste your time on algebraic "simplification" that actually makes matters more complicated.

Solution 4: $W(z) := 2\cdot Q((z-4)/3) - 1$ where $Q(t) = (1 + \bar{q})\cdot(q - t)/\big((1 + q)\cdot(1 - \bar{q}\cdot t)\big)$ for any constant q with $|q| < 1$. In particular, $W(z) = (5 - 2\cdot z)/3$ if $q = 0$; and $W(z) = (14/z - 11)/3$ if $q = -3/4$. Any other constants q with $|q| < 1$ (this is crucial) are eligible. Here is why:

Let $Q(t) := e^{\imath\theta}\cdot(t - q)/(1 - \bar{q}\cdot t)$ for constants θ and q to be chosen later. Since $|Q(e^{\imath\phi})| = 1$ for all real ϕ, so $Q(t)$ does map the t-plane's unit circle onto itself. $Q(q) = 0$, so Q maps the unit disk onto itself if and only if $|q| < 1$. (Q swaps the unit disk's interior with its exterior if $|q| > 1$.) For $|q| < 1$ let $W(z) := 2\cdot Q((z-4)/3) - 1$. Evidently $w = W(z)$ does map the disk $|z - 4| \le 3$ to the disk $|w + 1| \le 2$. Next we need $1 - W(1) = 0$. A little algebraic labor reveals that $1 - W(1) = 2 + 2\cdot e^{\imath\theta}\cdot(1 + q)/(1 + \bar{q})$, which forces us to choose θ so that

$$Q(t) = \big(-(1 + \bar{q})/(1 + q)\big)\cdot(t - q)/(1 - \bar{q}\cdot t) \quad \text{and then} \quad W(z) = 2\cdot Q((z-4)/3) - 1 .$$

5: Suppose $u(x, y)$ is harmonic everywhere in the Cartesian (x, y)-plane, and takes only positive values. What further constraint does this positivity impose upon $u(x, y)$, and why?

Solution 5: $u(x, y)$ is a positive constant. Here is why: Because $u(x, y)$ has no singularity in the finite plane, there is some *entire* (everywhere analytic, no finite singularities) function $f(z)$ with $u(x, y) = \mathrm{Real}(f(x + \imath\cdot y)) > 0$. Then $e^{-f(z)}$ is another entire function, and bounded too since $|e^{-f(z)}| < 1$. *Liouville*'s theorem forces $e^{-f(z)}$ to be a constant, whence $f(z)$ is a constant too, and so is its real part $u(x, y)$. (Liouville's theorem is the text's 2.4.8.)

6: Explain why no harmonic conjugate of $\log(x^2 + y^2)$ can have as its domain all of the finite (x, y)-plane except its origin $(0, 0)$.

Solution 6: Let $h(x, y)$ be any harmonic conjugate of $g(x, y) := \log(x^2 + y^2)$. Then some analytic function $f(z)$ has $f(x + \imath\cdot y) = g(x, y) + \imath\cdot h(x, y)$. The *Cauchy-Riemann* equations provide $f(z)$ with a derivative $f'(z)$ satisfying

$$f'(x + \imath\cdot y) = \partial g(x, y)/\partial x + \imath\cdot\partial h(x, y)/\partial x = \partial h(x, y)/\partial y - \imath\cdot\partial g(x, y)/\partial y .$$

It can be determined from $g(x, y)$ without explicit knowledge of $h(x, y)$ thus:

$$\partial g(x, y)/\partial x = 2x/(x^2 + y^2), \quad \partial g(x, y)/\partial y = 2y/(x^2 + y^2), \quad f'(x + \imath\cdot y) = 2(x - \imath\cdot y)/(x^2 + y^2) .$$

Evidently $f'(z) = 2/z$, whence follows $f(z) = 2\cdot\log(z) + 2\imath\cdot C = \log(|z|^2) + 2\imath\cdot(\mathrm{Arg}(z) + C)$ for some "Constant" C that is actually only locally constant: $\mathrm{Arg}(z) + C$ must jump by $\pm 2\pi$

when z crosses an arbitrarily drawn slit along any simple curve joining 0 to ∞. Otherwise $f(z)$ would have to be a multi-valued analytic function. Therefore the harmonic conjugate $h(x, y) = \mathrm{Imag}(f(x + \imath\cdot y))$ must also jump when (x, y) crosses that slit. In consequence $h(x, y)$ cannot be harmonic on the slit. (*Cf.* the text's problem #10 at the end of §2.5.)

7: Let "log" denote here the *Principal Value* whose domain has a slit along the negative real axis, and let $f(z) := \log(-(1 + z)/(99 + 20\cdot\imath))$. Its domain has its slit on the half-line whereon $z = -1 + (99 + 20\cdot\imath)\cdot\lambda$ for all $\lambda > 0$. What is the radius of convergence of the Taylor series expansion of $f(z)$ centered at $z = 0$?

Solution 7: The radius of convergence is 1 , not the shorter distance 20/101 from the center $z = 0$ to the slit in $f(z)$'s domain. The Taylor series of $F(z) = \log(1 + z) - \log(-99 - 20\cdot\imath)$ is the same as that of $f(z)$, though the functions have different domains, since their intersection where $f(z) = F(z)$ includes the center $z = 0$; and they have the same branch-point at $z = -1$.

8: Chaplygin's formula is $\overline{F} = \imath\cdot\rho\cdot\int_{\partial\mathbb{D}} \Phi'(z)^2\cdot dz/2$ to get the force F exerted by a "perfect" fluid of density ρ upon a body $\mathbb{D}$ when its boundary $\partial\mathbb{D}$ is a streamline of the fluid's flow, whose velocity potential is $\Phi(z)$. Compute the force F for a circulating flow past the unit disk given its velocity potential $\Phi(z) := (z + 1/z)/2 - \imath\cdot\gamma\cdot\log(z)$. Here real constant γ is proportional to the fluid's circulation.

Solution 8: $F = -\imath\cdot\rho\cdot\gamma\cdot\pi$. Here is why: $\Phi'(z) = \frac{1}{2} - \imath\cdot\gamma/z - \frac{1}{2}/z^2$, whence follows

$$\overline{F} = (\imath\cdot\rho/2)\cdot\int_{\partial\mathbb{D}} (\tfrac{1}{4} - \imath\cdot\gamma/z - (\gamma^2 + \tfrac{1}{4})/z^2 + \imath\cdot\gamma/z^3 + \tfrac{1}{4}/z^4)\cdot dz = (\imath\cdot\rho/2)\cdot(-\imath\cdot\gamma)\cdot 2\pi\cdot\imath = \imath\cdot\rho\cdot\gamma\cdot\pi\ .$$

Its complex conjugate is the force $F = -\imath\cdot\rho\cdot\gamma\cdot\pi$ as claimed.

9: Evaluate $K(\mu) := \int_0^{2\pi} dx/|1 + \mu\cdot e^{\imath x}|^2$ wherein $0 < \mu \neq 1$.

Solution 9: $K(\mu) = 2\pi/|1 - \mu^2|$. Here is why: Let $z := e^{\imath x}$ so that z runs once around the unit circle, which we denote by ©, as x runs from 0 to 2π . Then $dz = \imath\cdot z\cdot dx$, whence follows

$$K(\mu) = \int_0^{2\pi} dx/|1 + \mu\cdot e^{\imath x}| = -\imath\cdot\int_{©} dz/\big(z\cdot(1 + \mu\cdot z)\cdot(1 + \mu\cdot\bar{z})\big) = -\imath\cdot\int_{©} dz/\big((1 + \mu\cdot z)\cdot(z + \mu)\big) =$$

$$= -\imath\cdot\int_{©} (\ 1/(z + 1/\mu) - 1/(z + \mu)\ )\cdot dz/(\mu^2 - 1) =$$

$$= -\imath\cdot(\Delta \log(z + 1/\mu)|_{©} - \Delta \log(z + \mu)|_{©}\ )/(\mu^2 - 1)\ .$$

If $0 < \mu < 1$ then $\Delta \log(z + 1/\mu)|_{©} = 0$, $\Delta \log(z + \mu)|_{©} = 2\pi\cdot\imath$ and $K(\mu) = 2\pi/(1 - \mu^2)$.

If $\mu > 1$ then $\Delta \log(z + 1/\mu)|_{©} = 2\pi\cdot\imath$, $\Delta \log(z + \mu)|_{©} = 0$ and $K(\mu) = 2\pi/(\mu^2 - 1)$.

Either way, $K(\mu) = 2\pi/|1 - \mu^2|$ as claimed. (*Cf.* the text's Example 4.3.1 in §4.3 .)