Exercise 19: By integrating derivatives of a harmonic *function* g(x, y) we recover one of its harmonic conjugates h(x, y) and then set $f(x + ıy) := g(x, y) + ıh(x, y)$ to recover an analytic function $f(z)$. The recovery of an analytic *expression* f from a harmonic *expression* g that is also a real analytic function of each of its arguments can be achieved more easily by setting $f(z) := 2g((\bar{z}_o+z)/2,\ ı(\bar{z}_o–z)/2)$ for any $z_o = x_o + ıy_o$ inside the domain of g ; explain why. If this procedure fails to recover a *function* f analytic throughout the domain of g , explain why.

Explanation: A well-formed *expression* provides a formula to evaluate an analytic function by means of finitely many operations of addition, subtraction, multiplication, division, roots, log, cos, tanh, *etc*. For example, $f(z) := z^2 – ı/z$ leads to a formula $g(x, y) := x^2 – y^2 – y/(x^2 + y^2)$ for the harmonic function g that satisfies $f(x + ıy) = g(x, y) + ıh(x, y)$ when x and y are real. This g(x, y) is a formula also for a real analytic function of each of x and y when they are reinterpreted as complex variables. And after suitable complex expressions are substituted for x and y in the formula $2g((\bar{w}+z)/2,\ ı(\bar{w}–z)/2)$ it simplifies to $f(z)$ + (an expression in $\bar{w}$), which recovers f from g within a "constant" without integrating derivatives. Does this always work?

The complex conjugate of $f(x + ıy) := g(x, y) + ıh(x, y)$ is $\bar{f}(x + ıy) := g(x, y) – ıh(x, y)$ when x and y are real. This $\bar{f}(z)$ is generally not an analytic function of z because g and –h violate the Cauchy-Riemann conditions. However $Φ(z) := \bar{f}(\bar{z})$ *is* an analytic function of z because $Φ(x + ıy) = g(x, –y) – ıh(x, –y)$ for real x and y has real and imaginary parts g(x, –y) and –h(x, –y) that inherit satisfaction of the Cauchy-Riemann conditions from g(x, y) and h(x, y). For later reference note that the domain of Φ(z) is the complex conjugate of the domain of f.

We are given a harmonic g and wish to recover f (within a constant) without first constructing g 's harmonic conjugate h (within a constant). Eliminating h from foregoing formulas yields $2g(x, y) = f(x + ıy) + Φ(x – ıy)$. This relation among functions is also, presumably, a relation among expressions (formulas), and as such is a relation among formulas into which independent complex variables may be substituted for x and y because all three functions are analytic in their arguments no matter whether real or complex. An apt substitution is suggested by the equations $x = (\bar{z}+z)/2$ and $y = ı(\bar{z}–z)/2$ in which $\bar{z}$ is replaced by an independent complex variable $\bar{w}$, say. Thus $2g((\bar{w}+z)/2,\ ı(\bar{w}–z)/2) = f(z) + Φ(\bar{w}) = f(z) + \bar{f}(w)$ provided $\bar{w}$ is chosen in the domain of Φ, which puts w in the domain of f and of $\bar{f}$. The last equation delivers a formula for $f(z)$ within a constant when w is held constant. Almost.

This formula for f can malfunction in two ways. One way occurs when w does not lie in the domain of f ; for example when $g(x, y) = x^2 – y^2 – y/(x^2 + y^2)$ the choice $w := 0 ± ı0$ makes $2g((\bar{w}+z)/2,\ ı(\bar{w}–z)/2) = ∞$ instead of $z^2 – ı/z$ + (finite constant). This malfunction is predictable because g(x, y) misbehaves around (0, 0) in the (x, y)-plane, so $f(z)$ must be expected to have a singularity at $z = 0 ± ı0$ in the z-plane. The second kind of malfunction is more subtle.

If not simply connected, the domain in the (x, y)-plane whereon g(x, y) is harmonic may extend beyond the domain in the z-plane whereon $f(z)$ is differentiable. Try the Principal Value of $f(z) := \log(z)$ whose real part $g(x, y) = \log(x^2 + y^2)/2$ is harmonic everywhere but at (0, 0); its $2g((\bar{w}+z)/2,\ ı(\bar{w}–z)/2) = \log(\bar{w}·z)$ is discontinuous across the ray $z/w < 0$ instead of $z < 0$.

In general $2g((\bar{w}+z)/2,\ ı(\bar{w}–z)/2) – f(z)$ may take several "constant" values. Try $f(z) := √(2z)$.