The Players

Real variables are r, s, u, v, x, y, Ø, µ, ß, ρ ;
Complex variables are $t := r + \imath s$, $w := u + \imath v$, $z := x + \imath y$ where $\imath := \sqrt{(-1)}$.
Real functions are g, h, P, Q, X, Y ;
Complex functions are $f := g + \imath h$, p, q , usually.

Differentiable Functions

We suppose that complex $f(z) = f(x + \imath y) = g(x, y) + \imath h(x, y)$ and that g and h are real continuously differentiable functions of (x, y) in the sense that

$$dg(x, y) = g_{10}(x, y)\cdot dx + g_{01}(x, y)\cdot dy \quad \text{and} \quad dh(x, y) = h_{10}(x, y)\cdot dx + h_{01}(x, y)\cdot dy$$

with continuous partial derivatives $g_{10}(x, y) = \partial g(x, y)/\partial x$, ..., $h_{01}(x, y) = \partial h(x, y)/\partial y$. Here the differentials d… are to be determined by the Chain Rule as follows: Substitute arbitrary differentiable real functions X(ß) and Y(ß) of some real variable ß for x and y respectively; then every d… can validly be replaced by d…/dß in the equations above for dg and dh .

Is f a differentiable function of z ? The answer would already be "Yes" if, instead of complex variable and function, $\mathbf{z}$ and $\boldsymbol{f}$ were vectors in 2-dimensional real Euclidean space, say row-vectors $\mathbf{z} = [x, y]$ and $\boldsymbol{f} = [g, h]$. Then $d\boldsymbol{f} = [dg, dh]$ and $d\mathbf{z} = [dx, dy]$ would have to satisfy $d\boldsymbol{f}(z) = d\mathbf{z}\cdot\boldsymbol{f}'(\mathbf{z})$ for a 2-by-2 *Jacobian* matrix $\boldsymbol{f}'(\mathbf{z})$ whose elements were the four partial derivatives of g(x, y) and h(x, y) above; $\boldsymbol{f}' = \begin{bmatrix} g_{10} & h_{10} \\ g_{01} & h_{01} \end{bmatrix}$. But this $\boldsymbol{f}'(\mathbf{z})$ could not lie in the same space with $\mathbf{z}$ and $\boldsymbol{f}(\mathbf{z})$; to write " $d\boldsymbol{f}(z) = \boldsymbol{f}'(\mathbf{z})\cdot d\mathbf{z}$ " would be wrong because no 2-by-2 matrix $\boldsymbol{f}'(\mathbf{z})$ can premultiply a row-vector $d\mathbf{z}$.

Multiplication of complex variables is *commutative*; $w\cdot z = z\cdot w$. If f' is to be a complex function whose multiplication by other complex functions and variables is commutative too, the matrix $\boldsymbol{f}'$ cannot be an arbitrary matrix. We have already seen that special 2-by-2 matrices

$$W := \begin{bmatrix} u & v \\ -v & u \end{bmatrix}, \quad Z := \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \quad F := \begin{bmatrix} g & h \\ -h & g \end{bmatrix}, \quad \dots$$

are algebraically *isomorphic* with complex variables and functions $w = u + \imath v$, $z = x + \imath y$, $f = g + \imath h$, … respectively in so far as rational algebraic operations are concerned. That these matrices commute only with their own kind is easy to confirm, so matrix $\boldsymbol{f}'$ has to be a special 2-by-2 matrix too: $h_{01} = g_{10}$ and $g_{01} = -h_{10}$. The last two equations are the famous Cauchy-Riemann Equations, about which we have just deduced …

> If the complex function $f(z)$ of the complex variable z has a complex-valued derivative $f'(z)$ satisfying $df(z) = f'(z)\cdot dz$ for all complex dz, then $f(x + \imath y)$'s real and imaginary constituents $g(x, y) := \mathrm{Re}(f(x + \imath y))$ and $h(x, y) := \mathrm{Im}(f(x + \imath y))$ must satisfy the
> *Cauchy-Riemann Equations*: $\partial h(x, y)/\partial y = \partial g(x, y)/\partial x$ and $\partial g(x, y)/\partial y = -\partial h(x, y)/\partial x$;
> and then $f'(x + \imath y) = \partial g(x, y)/\partial x + \imath\partial h(x, y)/\partial x = \partial h(x, y)/\partial y - \imath\partial g(x, y)/\partial y$.

Example: The function $|z|^2 = \bar{z}\cdot z$ has no complex derivative at $z \neq 0$ because then the real and imaginary parts of $|x + \imath y|^2 = (x^2 + y^2) + \imath\cdot 0$ violate the Cauchy-Riemann Equations. $|z|^2$ is a differentiable function of *real* variables x and y since $d(|z|^2) = \bar{z}\cdot dz + z\cdot d\bar{z} = 2x\cdot dx + 2y\cdot dy$.

Exercise 1: At every complex $z \ne 0$ the function $1/z = \bar{z}/|z|^2$ has a complex derivative $-1/z^2$, so the real and imaginary parts of $1/(x + \imath y) = x/(x^2 + y^2) + \imath\cdot(-y/(x^2 + y^2))$ must satisfy the Cauchy-Riemann Equations, and they do; check them out.

Exercise 2: Suppose $f(z)$ is differentiable over an open domain in the z-plane mapped by the assignment $w := f(z)$ to some region in the w-plane. Show that the images $f(Ç)$ and $f(ç)$ in the w-plane of any two smooth curves $Ç$ and $ç$ in the z-plane have the same angles of intersection except if $Ç$ and $ç$ intersect at a *critical point* z where $f'(z) = 0$. This is why such a map is called a *Conformal* map. Hint: What do orthogonal 2-by-2 matrices do?

How does a given expression for $f(z)$ get turned into an expression for $f'(z)$?

If $f(z)$ is an *algebraic* function, the rules for symbolic differentiation turn out to be the same for complex as for real expressions. The first rule worth knowing is that the derivative $f'(z)$ is the limit, as $w \to z$, of the …

Divided Difference $f^\dagger(\{z, w\}) := (f(z) - f(w))/(z-w)$ simplified symbolically,

which, as we shall see, simplifies symbolically to …

a polynomial in z and w if $f(z)$ is a polynomial in z ,
a rational function of z and w if $f(z)$ is a rational function of z , or
an algebraic function of z and w if $f(z)$ is a continuous algebraic function of z ,

after the division by $(z-w)$ has been carried out. *After* that simplification, $f'(z) = f^\dagger(\{z, z\})$.

Note: $f^\dagger(\{z, w\})$ is a function of unordered *pair* $\{z, w\}$, so that $f^\dagger(\{z, w\}) = f^\dagger(\{w, z\})$.

There is no standard notation for $f^\dagger(\{z, w\})$; other authors use $[z, w]f$ or $f[z, w]$ or $\Delta f(\{z, w\})$ or … .

Example: For any integer $N \ge 0$ the divided difference of $f(z) := z^N$ simplifies to

$$f^\dagger(\{z, w\}) = \sum_{1 \le k \le N} z^{k-1}\cdot w^{N-k} ,$$

after which $f'(z) = f^\dagger(\{z, z\}) = N\cdot z^{N-1}$, the same for complex as for real z .

"The same for complex as for real" generalizes to arbitrary rational functions $f(z)$ because, as is easily verified

if $f(z) = p(z) \pm q(z)$ then $f^\dagger(\{z, w\}) = p^\dagger(\{z, w\}) \pm q^\dagger(\{z, w\})$ respectively, and
if $f(z) = p(z)\cdot q(z)$ then $f^\dagger(\{z, w\}) = p^\dagger(\{z, w\})\cdot q(z) + p(w)\cdot q^\dagger(\{z, w\})$, and
if $f(z) = p(z)/q(z)$ then $f^\dagger(\{z, w\}) = \big(p^\dagger(\{z, w\})\cdot q(w) - p(w)\cdot q^\dagger(\{z, w\}) \big)/\big(q(z)\cdot q(w) \big)$.

The latter two formulas have alternatives generated by the identity $f^\dagger(\{z, w\}) = f^\dagger(\{w, z\})$.

The second rule worth knowing is the …

Chain Rule: If $p(w)$ and $q(z)$ are complex differentiable functions of complex arguments, then $f(z) := p(q(z))$ has a complex derivative $f'(z) = p'(q(z))\cdot q'(z)$.

This follows directly from the Chain Rule for differentiable vector-valued functions of vector arguments; first treat z, q, p and f as 2-vectors, and then convert derivatives from special 2-by-2 matrices back to their complex form. Another way to go is the Divided Differences' …

Chain Rule: If $f(z) := p(q(z))$ then $f^\dagger(z, w) = p^\dagger(\{q(z), q(w)\})\cdot q^\dagger(\{z, w\})$.

Example: If $f(z) := (3z + 1/z)^2$, then $f^\dagger(z, w) = \big((3z+1/z) + (3w+1/w)\big)\cdot\big(3 - 1/(z\cdot w)\big)$, and then $f'(z) = 2(3z + 1/z)\cdot(3 - 1/z^2)$, since $p(t) := t^2$ has $p^\dagger(\{s,t\}) = s+t$.

Analogous to the partial derivatives $\partial p(z, t)/\partial z$ and $\partial p(z, t)/\partial t$ of a differentiable function $p(z, t)$ are its partial divided differences $p^\dagger(\{z,w\}, t) := (\,p(z,t) - p(w,t)\,)/(z-w)$, simplified symbolically to eliminate $\ldots/(z-w)$, and similarly $p^\dagger(z, \{w,t\}) = (\,p(z,w) - p(z,t)\,)/(w-t)$. Once again, if $p(z, t)$ is a polynomial in z and t , so are its partial derivatives and divided differences. Likewise if p is a rational function.

To deal with algebraic functions we must introduce *Implicit Divided Differencing*:

If $f(z)$ is a root of the equation $p(z, f(z)) = 0$ then

$$f^\dagger(\{z,w\}) = -p^\dagger(\{z,w\}, f(z))/p^\dagger(w, \{f(z),f(w)\}) .$$

This follows by elementary algebra from $p(z, f(z)) = p(w, f(w))$ and the definitions of $\ldots^\dagger$. Then it is tempting to assume f is continuous, let $f(w) \to f(z)$ as $w \to z$, and deduce that

$$f'(z) = -\partial p(z, t)/\partial z \,/\, \partial p(z, t)/\partial t \quad \text{evaluated at } t = f(z) .$$

Algebraic functions f aren't quite so simple, as we'll see after we discuss Implicit Functions.

Example: When $p(z, t) := t^2 - z$, a particular root of $p(z, f(z)) = 0$ is $f(z) := \sqrt{z}$, and this yields $f^\dagger(\{z, w\}) = 1/(\sqrt{z} + \sqrt{w})$. Since $\sqrt{z}$ is continuous except as z crosses a slit along the negative real axis in the z-plane, $d(\sqrt{z})/dz = f'(z) = \lim_{w\to z} f^\dagger(\{z, w\}) = f^\dagger(\{z, z\}) = 1/(2\sqrt{z})$ except on the slit. Continuity is essential here; had we taken $f(z) = \sqrt{z}$ and $f(w) = -\sqrt{w}$ as roots of $p(z, f(z)) = 0$ and $p(w, f(w)) = 0$ respectively, we would have obtained a divided difference $(\,f(z) - f(w)\,)/(z-w) = 1/(\sqrt{z} - \sqrt{w})$ with ∞ instead of the correct derivative for its limit. We mustn't make this mistake when we discuss Implicit Functions.

Just as the derivative is the limit (when it exists) of a divided difference, the divided difference is an average of the derivative (when it is exists and is continuous) thus:

$$\text{Hermite's formulation:} \quad f^\dagger(\{z, w\}) := \int_0^1 f'(w + (z-w)\mu)\,d\mu .$$

In other words, $f^\dagger(\{z, w\})$ is the uniformly weighted average of f' on the line segment joining z and w . Hermite's formulation needs no division by $(z-w)$, so it "works" also for (perhaps vector-valued) functions $\boldsymbol{f}(\mathbf{z})$ of vector arguments $\mathbf{z}$, for which we find …

Exercise 3: Deduce from Hermite's formulation that

$$\boldsymbol{f}(\mathbf{z}) - \boldsymbol{f}(\mathbf{w}) = \boldsymbol{f}^\dagger(\{\mathbf{z}, \mathbf{w}\})\cdot(\mathbf{z}-\mathbf{w}) , \quad \boldsymbol{f}^\dagger(\{\mathbf{z}, \mathbf{w}\}) = \boldsymbol{f}^\dagger(\{\mathbf{w}, \mathbf{z}\}) , \quad \text{and} \quad \boldsymbol{f}^\dagger(\{\mathbf{z}, \mathbf{z}\}) = \boldsymbol{f}'(\mathbf{z}) .$$

Then show that the divided difference of a polynomial function of a vector variable's elements is also a polynomial function of the vector's elements, but Hermite's divided difference of a rational function of a vector variable's elements can be a non-rational function of them, alas.

This is why derivatives drove divided differences out of fashion: $\boldsymbol{f}^\dagger$ has more variables than $\boldsymbol{f}'$ has, and Hermite's $\boldsymbol{f}^\dagger$ can be transcendental when $\boldsymbol{f}'$ is still rational or algebraic. Other ways to define $\boldsymbol{f}^\dagger$ exist that remain rational or algebraic respectively and satisfy Exercise 3's equations, but they depend upon the coordinate system in the space of $\boldsymbol{f}$'s argument, and lack properties that will be revealed in the following digression.

Digression: Interpolation with Higher-Order Divided Differences

Many *Numerical Analysis* texts and all texts about *Finite Differences* devote at least a chapter to divided differences of higher than first order. Here they will be surveyed only enough to convey a sense of their utility. Proofs will be omitted during this digression.

Alas, no fully satisfactory notation exists for higher-order divided differences, so the literature does not agree upon one. For $n = 0, 1, 2, 3, \ldots$ let us write $f^{[n]}(x)$ for the n^{th} derivative of $f(x)$ at argument x, and write $f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\})$ for the n^{th} divided difference of $f(x)$ over argument $(n+1)$-tuple $\{x_0, x_1, \ldots, x_n\}$. Yes, $f^{[0]} = f^{\dagger 0\dagger} = f$, $f^{[1]} = f'$ and $f^{\dagger 1\dagger} = f^{\dagger}$. Here for $n > 0$ is Hermite's formulation: $n!\cdot f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\})$ is the uniformly weighted average of $f^{[n]}(x)$ as x ranges over a *simplex* whose $n+1$ vertices are $x_0, x_1, \ldots, x_n$;

$$f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\}) := \int_0^1\int_0^{q_1}\int_0^{q_2}\cdots\int_0^{q_{n-1}} f^{[n]}\Big(x_0 + \sum_{j=1}^{n} q_j(x_j - x_{j-1})\Big)dq_n\ldots dq_3dq_2dq_1 .$$

The simplex *degenerates* (collapses) if the vectors $x_j - x_0$ for $j = 1, 2, \ldots, n$ are linearly dependent, but the foregoing integral remains valid and, most important, independent of the order of the x_j's. For example, when $n = 2$ the simplex is a triangle and, as you should verify by manipulating the integral, $f^{\dagger\dagger}(\{x, y, z\}) := f^{\dagger 2\dagger}(\{x, y, z\}) = f^{\dagger\dagger}(\{y, z, x\}) = f^{\dagger\dagger}(\{y, x, z\})$, and so on. And $f^{\dagger\dagger}(\{x, x, x\}) = f''(x)/2$ just as $f^{\dagger n\dagger}(\{x, x, \ldots, x\}) = f^{[n]}(x)/n!$ in general.

Divided differences supply remainders to *Taylor*'s formula: if $z = x+h$ then $f(z) = \ldots$

$$f(x) + f'(x)\cdot h + f''(x)\cdot h^2/2 + f^{[3]}(x)\cdot h^3/3! + \ldots + f^{[n-1]}(x)\cdot h^{n-1}/(n-1)! + f^{\dagger n\dagger}(\{x, x, \ldots, x, z\})\cdot h^n$$

in which the $(n+1)$-tuple $\{x, x, \ldots, x, z\}$ has x repeated n times followed by z. This formula is valid for vector arguments x and z provided " h^k " is interpreted to mean k repetitions of the vector $h = z-x$ as the arguments for the k-linear operators $f^{[k]}$ and $f^{\dagger k\dagger}$, which act linearly on each of k vector arguments thus: $f^{[k]}(x)\cdot v_1\cdot v_2\cdot\ldots\cdot v_k$ is a linear function of each vector v_j separately and, because of the Clairault/Schwartz theorem that says the order in which multiple differentiations are performed doesn't matter if derivatives are continuous, the order of vectors v_j after $f^{[k]}(x)$ doesn't matter either.

Divided differences allow Taylor's formula to be generalized to *Newton's Divided Difference* formula, of which the following instance is simplified to dispense with subscripts: $f(z) = \ldots$

$$f(y) + f^{\dagger}(\{x,y\})\cdot(z-y) + f^{\dagger 2\dagger}(\{w,x,y\})\cdot(z-x)\cdot(z-y) + f^{\dagger 3\dagger}(\{v,w,x,y\})\cdot(z-w)\cdot(z-x)\cdot(z-y) +$$
$$+ f^{\dagger 4\dagger}(\{v,w,x,y,z\})\cdot(z-v)\cdot(z-w)\cdot(z-x)\cdot(z-y) .$$

If the last "remainder" term in this formula (and in Taylor's) is small enough, as it is if $f^{[4]}$ is small enough or if z is close enough to v, w, x and y, then f may well be approximated by the polynomial function of its argument consisting of the previous terms; they constitute an *Interpolating* polynomial in z which matches the values $f(z)$ takes when $z = y$, $z = x$, $z = w$ and $z = v$. If some of these arguments v, w, x, y coincide, certain of the polynomial's derivatives match f's at those repeated arguments.

Note: Newton's and Taylor's formulas work for non-scalar (vector) arguments v, w, x, y, z only with Hermite's formulation (Genocchi used it too) of divided differences, and then the polynomial's degree need not be minimal.

$n!\cdot f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\})$ is an average which must take a value inside the convex hull of the values taken by $f^{[n]}(x)$ as x ranges over the convex hull of the points $x_0, x_1, \ldots$ and x_n, but that average value need not be a value taken by $f^{[n]}$ anywhere. When f and its arguments v, w, x, y, z are restricted to scalars, real or complex, much more can be said. In the real case, if $f^{[n]}$ is continuous it must take its average value at some place ξ inside any interval containing all of $x_0, x_1, \ldots, x_n$, and then $f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\}) = f^{[n]}(\xi)/n!$. This form for the remainder occurs frequently in the literature. In general, any expression for $f^{[n]}$ that yields bounds for its values over some domain provides a bound for the difference between f and an interpolating polynomial of degree less than n over that domain .

When f is a complex analytic function of a complex scalar variable, though f may be real if its argument is real, $f^{\dagger n\dagger}$ can be bounded without computing $f^{[n]}$ provided $|f|$ can be bounded over a suitable closed contour C in the complex plane. C must enclose no singularity of f and yet enclose all the points $x_0, x_1, \ldots$ and x_n. Let $ß(z) := \prod_0^n (z-x_j)$; then it turns out that $2\imath\pi\cdot f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\}) = \int_C f(z)dz/ß(z)$. The contour integral implies that, if every $|z-x_j| > \mu$ while z runs on C whereon $|f(z)| < M$, then

$$2\pi\cdot|f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\})| < M\cdot\text{length}(\,C\,)/\mu^{n+1}\ .$$

We prefer that this bound be small in order that an interpolating polynomial of degree less than n approximate f well, but the bound cannot be made arbitrarily small because M·length(C) generally grows ultimately rather faster than μ^{n+1} as μ increases, so some skill is needed to choose C well enough to produce a bound about as small as possible.

The contour integral produces an explicit formula for $f^{\dagger n\dagger}$ valid even if f is not analytic; in the simple case that all the points $x_0, x_1, \ldots$ and x_n are distinct, the formula is simply $f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\}) = \sum_j f(x_j)/ß'(x_j)$ in which $ß'(x_j)$ is the product of all n differences $(x_j - x_k)$ with $k \neq j$. In other words, $f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\})$ can be computed from the values f takes at distinct scalar arguments without any recourse to derivatives. However, the foregoing sum is rarely a satisfactory way to compute $f^{\dagger n\dagger}$ numerically. Usually better is

$$f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\}) = \left(f^{\dagger n-1\dagger}(\{x_0, x_1, \ldots, x_{n-1}\}) - f^{\dagger n-1\dagger}(\{x_1, x_2, \ldots, x_n\})\right)/(x_0 - x_n\}) ,$$

especially when points $x_0, x_1, \ldots$ and x_n are in monotonic order, exploiting the observation that high-order divided differences are divided differences of lower-order divided differences. That recurrence generalizes to produce a *Confluent* Divided Difference $f^{\dagger n\dagger}(\{x_0, x_1, \ldots, x_n\})$ when some of its arguments coincide; see Commun. Assoc. Comp. Mach. **6** (1963) pp. 164-5.

Exercise 4: This example shows why Newton's formula, used often to interpolate functions of scalar arguments, is almost never used with vector arguments. In the (x, y)-plane let rectangle R be given with diagonally opposite vertices at (0, 0) and (X, Y) . What is the expected degree of the polynomial that Newton's formula would produce to interpolate $f(x, y)$ at the four vertices of R ? The same interpolation is accomplished by the *Bilinear* polynomial $p(x, y) := f(0, 0) + x\cdot f^{\dagger}(\{0,X\}, 0) + y\cdot f^{\dagger}(0, \{0,Y\}) + x\cdot y\cdot f^{\dagger\dagger}(\{0,X\}, \{0,Y\})$. If (x, y) lies in R inside which f has continuous second derivatives, show that $f(x, y) - p(x, y) = x\cdot(x-X)\partial^2 f/\partial x^2 + y\cdot(y-Y)\partial^2 f/\partial y^2$ for derivatives each evaluated somewhere in R .

Implicit Functions

It is not obvious that every nontrivial polynomial equation $p(z, t) = 0$ must have roots t, much less that they can be chosen in a way that makes them continuous functions of z. The existence of as many roots $t(z)$ as the degree in t of $p(z, t)$ is a famous theorem of C.F. Gauss that we shall prove easily later. Their continuity is a theorem infamous for proofs that hide the difficulty of identifying correctly each of a number of single-valued functions $t(z)$ whose values may coalesce at critical points z where the equation has multiple roots.

For example, what are the roots t of $t^3 - z = 0$ if $z \neq 0$? The *Principal Cube Root* , often written simply " $z^{1/3}$ ", is the root t with $-\pi/3 < \arg(t) \le \pi/3$; if $z = r\cdot e^{\imath Ø}$ with $r \ge 0$ and $-\pi < Ø \le \pi$ then this $z^{1/3} = \sqrt[3]{r}\cdot e^{\imath Ø/3}$. It is discontinuous across the z-plane's negative real axis ($Ø = \pm\pi$) whereon it takes non-real values different from the *Near-Real Cube Root*, best denoted by " $\sqrt[3]{z}$ ", which is the root t with $-1/\sqrt{3} < \mathrm{Im}(t)/\mathrm{Re}(t) \le 1/\sqrt{3}$. On all of the real axis this $\sqrt[3]{x}$ is real and continuous with $\mathrm{sign}(\sqrt[3]{x}) = \mathrm{sign}(x)$. Elsewhere $\sqrt[3]{z}$ is discontinuous only across the imaginary axis. The two definitions of cube root disagree in the left half-plane; there $\sqrt[3]{z} = -(-z)^{1/3}$. Which definition is better? Neither. Consider the next example …

What are the roots t of $t^3 + 3z\cdot t - 2z = 0$ if $z \neq 0$? A widely used book tenders the formula " $t = z^{1/3}\cdot((1 + \sqrt{1+z})^{1/3} + (1 - \sqrt{1+z})^{1/3})$ " without mentioning that its three cube roots must be chosen in a correlated way. Principal Cube Roots for all three are never correct choices; …

Exercise 5: Explain why.

Near-Real Cube Roots for all three choices are better but imperfect; if the book's formula read " $t(z) = \sqrt[3]{z}\cdot(\sqrt[3]{(1 + \sqrt{1+z})} + \sqrt[3]{(1 - \sqrt{1+z})})$ " it would be wrong when $-\mathrm{Im}(z/2)^2 \le \mathrm{Re}(z) < 0$, *i.e*. between the imaginary axis and a parabola in the left half of the z-plane. This formula's three cube roots are chosen correctly only when the one chosen for z is the negative of the other two's product; otherwise this formula's $t(z)$ dissatisfies the equation $t^3 + 3z\cdot t - 2z = 0$.

Exercise 6: Explain why. Show for $z \neq 0$ that a correct formula (and perhaps the simplest) is
$t(z) = q - z/q$ for each choice q of one of the three cube roots of $(1 + \sqrt{1+z})\cdot z$.
Show that replacing $\sqrt{1+z}$ by $-\sqrt{1+z}$ merely permutes the three roots $t(z)$.
As $z \to \infty$ one of the three roots $t(z)$ approaches a finite limit; what is it?

But no choice of cube root for q makes this root $t(z)$ continuous over the whole z-plane. If q is the Principal Cube Root, t is discontinuous as z crosses either the line segment $-1 < z < 0$ or an approximately hyperbolic curve in the left half-plane on which $z = x+\imath y$ satisfies the equation $(3x^2 - y^2)^2 = -8x(x^2 + y^2)$. If q is the Near-Real Cube Root, t is discontinuous across another approximately hyperbolic curve in the right half-plane; this curve's equation is $(y(y^2 - 3x^2))^2 = x(y^4 - x^4)$.

In short, no formula for a root $t(z)$ of the equation $t^3 + 3z\cdot t - 2z = 0$ can be continuous on all the z-plane; every formula jumps as z crosses some curve joining the point 0 to at least one of -1 and ∞ . These three points are the equation's critical points where it has multiple roots, triple at $z = 0$, double at $z = -1$ and ∞ . As z moves around a critical point the formula for a root would have to become multi-valued if it stayed continuous. To see why, …

Exercise 7: Use the formula in Exercise 6 to trace the behavior of all three roots $t(z)$ as z traverses a tiny circle around a critical point, and watch two or three of the roots swap places as if playing *Musical Chairs*.

And yet, wherever a root $t(z)$ of the equation is *Simple* (*i.e.* non-multiple) it can be obtained from a formula that is continuous in some open region (perhaps small) in the z-plane. This is a harbinger of what happens generally to roots $t(z)$ of an analytic equation $p(z, t) = 0$. The simple roots' local continuity is a consequence of a general …

Implicit Function Theorem: If $p(z_o, t_o) = 0$, and if $p(z, t)$ is continuously differentiable in a neighborhood of $z = z_o$ and $t = t_o$, and if $\partial p(z_o, t)/\partial t \neq 0$ at $t = t_o$ (so t_o is a simple root of the equation $p(z_o, t_o) = 0$), then throughout some (maybe smaller) neighborhood of $z = z_o$ a continuously differentiable function $f(z)$ exists satisfying $p(z, f(z)) = 0$ and $f(z_o) = t_o$, and the derivative of f is $f'(z) = -\partial p(z, t)/\partial z \,/\, \partial p(z, t)/\partial t$ evaluated at $t = f(z)$.

Proof: You may have seen a theorem like this already for vector-valued functions of vector arguments, though then the place of $\ldots/\,\partial p(z, t)/\partial t$ was taken by $(\partial \mathbf{p}(z, \mathbf{t})/\partial \mathbf{t})^{-1}\cdot\ldots$ for a nonsingular instead of nonzero derivative. The following proof for complex variables is similar.

The easy part of the proof is the formula for $f'(z)$, which will follow from …

Implicit Differentiation : $0 = dp(z, f(z))/dz = \partial p(z, f(z))/\partial z + \partial p(z, t)/\partial t\big|_{t=f(z)}\cdot f'(z)$,

which will follow in turn from the formula for Implicit Divided Differencing above. The hard part is proving that a continuous $f(z)$ exists in some neighborhood of $z = z_o$. To simplify the proof's notation, let $q(t) := p(z, t)$ for some fixed z so close to z_o that $|q(t)|$ is very tiny for all t close enough to $t_o := f(z_o)$; later we shall choose "so close" and "close enough" to ensure that the equation $q(t) = 0$ has just one root $t = f(z)$ that close to $f(z_o)$. The needed closeness depends upon how wildly $q'(t) = \partial p(z, t)/\partial t$ varies, since it must be prevented from varying too much compared with $q'(t_o)$, which must be nonzero if z is close enough to z_o because then $q'(t_o)$ is very near the given nonzero $\partial p(z_o, t)/\partial t$ at $t = t_o$. These requirements come together in an important …

Lemma 1: If a positive constant μ can be found to satisfy $\mu > |q(t_o)/q^\dagger(\{t_o,t\})|$ whenever $|t-t_o| \le \mu$ then at least one root t of the equation $q(t) = 0$ also satisfies $|t-t_o| < \mu$. No more than one such root t exists if $q^\dagger(\{w, t\}) \neq 0$ too whenever both $|t-t_o| < \mu$ and $|w-t_o| < \mu$.

Proof: When two such roots exist, say $q(t) = q(w) = 0$ but $w \neq t$, then $q^\dagger(\{w, t\}) = 0$, which contradicts the second hypothesis. Therefore the lemma's nontrivial part is the inference from the first hypothesis that at least one root t exists. To this end define $Q(t) := t - q(t)/q^\dagger(\{t_o,t\})$. Since the divisor cannot vanish, $Q(t)$ is continuous throughout the closed disk $|t-t_o| \le \mu$. And $Q(t)-t_o = -q(t_o)/q^\dagger(\{t_o,t\})$, so $|Q(t)-t_o| < \mu$ throughout the disk; in other words, Q is a continuous map of the closed disk into itself. By Brouwer's *Fixed-Point Theorem*, Q must have a fixed-point $t = Q(t)$ in that disk; this is the root t we seek. End of Lemma 1's proof.

Readers who are still uncomfortable with the divided difference $q^\dagger$ can resort to a very similar lemma stated exclusively in terms of the derivative q' instead, though its proof is longer:

Lemma 2: If a positive constant μ can be found to satisfy $\mu > |q(t_o)/q'(t)|$ whenever t lies in the disk $|t-t_o| < \mu$ then at least one root t of the equation $q(t) = 0$ also lies inside that disk. No more than one such root exists if also $|1 - q'(t)/q'(t_o)| < 1$ whenever t is inside that disk.

Proof: We shall construct a *trajectory* $t = T(s)$ by solving an *initial value problem*

$$T(0) = t_o \quad \text{and} \quad dT/ds = -q(T)/q'(T) \quad \text{for} \quad 0 \le s \le S .$$

This differential equation must have at least one complex/vector solution $T(s)$ for all real s in some interval of sufficiently small positive width S because $-q(t)/q'(t)$ is a continuous function so long as t stays inside the disk wherein $q'(t) \ne 0$; see *Peano's Existence Theorem* in a textbook about *Ordinary Differential Equations*. And so long as $T(S)$ is inside the disk, S can grow a little. How big can the width S grow and still have the trajectory (graph of) T strictly inside the disk $|T-t_o| < \mu$? We are about to find that S can grow arbitrarily big.

First let S be any positive value for which the trajectory $T(s)$ stays inside the disk throughout $0 \le s < S$. Along the trajectory we find $dq(T(s))/ds = q'(T(s))\cdot T'(s) = -q(T(s))$, whence follows that $q(T(s)) = e^{-s}\cdot q(t_o)$. Then the length of the trajectory from $s = 0$ to $s = S$ must be

$$\int_o^S |dT(s)/ds|\cdot ds = \int_o^S |q(T(s))/q'(T(s))|\cdot ds = \int_o^S |q(t_o)/q'(T(s))|\cdot e^{-s}\cdot ds < \int_o^S \mu\cdot e^{-s}\cdot ds < \mu .$$

Evidently $T(S)$ lies strictly inside the disk for every positive S . Let $S \to +\infty$ to find

$$\int_o^{+\infty} |dT(s)/ds|\cdot ds = \int_o^{+\infty} |q(t_o)/q'(T(s))|\cdot e^{-s}\cdot ds < \int_o^{+\infty} \mu\cdot e^{-s}\cdot ds = \mu .$$

Therefore $T(+\infty)$ lies strictly inside the disk too. And $q(T(+\infty)) = 0$. In other words, the trajectory $t = T(s)$ ends strictly inside the disk at a root $t := T(+\infty)$ of the equation $q(t) = 0$.

Now we know the disk contains at least one root; can it contain two if also $|1 - q'(t)/q'(t_o)| < 1$ whenever t is inside that disk? No; in fact now $q(t)$ can take no value, zero or not, more than once inside that disk. Otherwise we could find T inside that disk too with $q(T) = q(t)$ but $T \ne t$, in which case we would also find

$$1 = |1 - q^\dagger(\{T, t\})/q'(t_o)| = |\int_o^1 (1 - q'(t + (T-t)s)/q'(t_o))ds| \le \int_o^1 |1 - q'(t + (T-t)s)/q'(t_o)|ds < 1 .$$

This contradiction ends the proof of Lemma 2.

Lemmas 1 and 2 provide ways to confirm that a root t of $q(t) = 0$ lies within a distance μ of t_o , but no way to find t_o nor μ . One way to seek μ , given q and t_o , is to choose an M somewhat bigger than $|q(t_o)/q'(t_o)|$ and find an overestimate μ of $\max_{|t-t_o|\le M} |q(t_o)/q'(t)|$ or of $\max_{|t-t_o|\le M} |q(t_o)/q^\dagger(\{t_o,t\})|$, hoping that this $\mu \le M$. But it need not be so. …

Exercise 8: For $q(t) := t^2 - 1$ and $t_o := 4$ show that no μ can be found to work in Lemma 2, but Lemma 1 works if $3 < \mu < 4$. For $q(t) := \exp(\imath t) + 1$ and $t_o := 3.13$ show that $\mu = 2\pi$ in Lemma 2 establishes the existence of a root t satisfying $|t-t_o| < \mu$, but not its uniqueness. Show that the second hypothesis of Lemma 2 implies the second of Lemma 1; and if $q(t)$ is a real function of a real argument t , and if t_o is real and the "disk" is an interval in which a real root t may lie , then show the first hypothesis of Lemma 2 implies the first of Lemma 1.

Back to the proof of the Implicit Function Theorem. Given that $p(z_o, t_o) = 0$, and for any given sufficiently tiny tolerance $\mu > 0$, we seek a tolerance ß so small that, for every z in the disk $|z - z_o| < ß$, the equation $p(z, t) = 0$ has just one root t in the disk $|t - t_o| < \mu$. Just one such root's existence is guaranteed by the lemmas provided $q(t) := p(z, t)$ satisfies either

(L1): $|q(t_o)|/|q^\dagger(\{t_o,t\})| < \mu$ and $q^\dagger(\{t, w\}) \neq 0$ whenever both $|t-t_o| < \mu$ and $|w-t_o| < \mu$,

or

(L2): $|q(t_o)/q'(t)| < \mu$ and $|1 - q'(t)/q'(t_o)| < 1$ whenever $|t-t_o| < \mu$.

Now, as $z \to z_o$ and $w \to t_o$ and $t \to t_o$, we find

(i) $q^\dagger(\{t, w\}) = p^\dagger(z, \{t,w\}) \to \partial p(z_o, t_o)/\partial t \neq 0$,

(ii) $q(t_o)/((z-z_o)\cdot q^\dagger(\{t_o,t\})) = p^\dagger(\{z,z_o\}, t_o)/p^\dagger(z, \{t,t_o\}) \to \partial p(z_o, t_o)/\partial z/\partial p(z_o, t_o)/\partial t$,

(iii) $q(t_o)/((z-z_o)\cdot q'(t)) = p^\dagger(\{z,z_o\}, t_o)/\partial p(z, t_o)/\partial t \to \partial p(z_o, t_o)/\partial z/\partial p(z_o, t_o)/\partial t$, and

(iv) $1 - |1 - q'(t)/q'(t_o)| = 1 - |1 - \partial p(z, t)/\partial t/\partial p(z_o, t_o)/\partial t| \to 1$.

Choose any positive constant $K > |\partial p(z_o, t_o)/\partial z/\partial p(z_o, t_o)/\partial t|$. By continuity, tiny positive tolerances $ß_1$, $ß_2$, $ß_3$, $ß_4$, μ_1, μ_2, μ_3, μ_4 must exist such that

(i) $q^\dagger(\{t, w\}) = p^\dagger(z, \{t,w\}) \neq 0$ if $|z-z_o| < ß_1$ & $|t-t_o| < \mu_1$ & $|w-t_o| \le \mu_1$,

(ii) $|q(t_o)/((z-z_o)\cdot q^\dagger(\{t_o,t\}))| < K$ if $|z-z_o| < ß_2$ & $|t-t_o| < \mu_2$,

(iii) $|q(t_o)/((z-z_o)\cdot q'(t))| < K$ if $|z-z_o| < ß_3$ & $|t - t_o| \le \mu_3$, and

(iv) $|1-q'(t)/q'(t_o)| = |1 - \partial p(z, t)/\partial t/\partial p(z_o, t_o)/\partial t| < 1$ if $|z-z_o| < ß_4$ & $|t - t_o| \le \mu_4$.

Given any positive $\mu < \min_j \{\mu_j\}$, keeping $|z-z_o| < ß := \min\{ \min_j \{ß_j\}, \mu/K \}$ ensures that μ satisfies both lemmas' requirements, thus guaranteeing that just one root t exists in $|t-t_o| < \mu$. End of proof of Implicit Function Theorem.

This theorem's proof works for arbitrary analytic functions $p(z, t)$ of two complex variables, not just polynomials, but it is not the best theorem. It establishes the differentiability of simple roots of analytic equations without revealing what happens when roots coalesce. In fact, roots remain continuous, if not differentiable, where they coalesce; we shall prove this when we come to Rouché's theorem. For now we are content with the Implicit Function Theorem's assurance that every algebraic expression has a continuous divided difference and a derivative, both also algebraic expressions, except at *Branch Points* (where roots coalesce) and across somewhat arbitrary *Slits* introduced to make the expression single-valued. Like rational expressions, algebraic expressions can have *Poles* where they take infinite values; but poles need not detract from continuity and differentiability if the expressions are construed as maps from the *Riemann Sphere* to itself. (For this sphere see the notes on Möbius Transformations.)

Elementary Transcendental Functions

Non-algebraic analytic functions are called "Transcendental". The *Elementary Transcendental* functions arise out of algebraic operations upon exp(z) and ln(z). The latter's multiplicity of values, each differing from others by integer multiples of $2\pi\imath$, is suppressed by a notation that assigns one *Principal Value* to " ln(z) " in some contexts and, if done conscientiously, uses another notation like " Ln(z) " for the multi-valued version. One text, *Complex Variables and Applications* 6th. ed. (1996) by Brown & Churchill (McGraw-Hill), does just the opposite.

Formulas Defining Principal Values of Inverse Elementary Functions

Complex $z = x + ıy$ has $\bar{z} := x - ıy$ for real $\mathrm{Re}(z) := x$ and $\mathrm{Im}(z) := y$. $ı^2 = -1$.

$$|x + ıy| := \sqrt{(x^2 + y^2)} \; ; \quad \text{in other words} \quad |z| := \sqrt{(z\cdot\bar{z})} \ge 0 \, .$$

$$\arg(x + ıy) := 2\arctan(y/(x + |x + ıy|)) \quad \text{if } y \ne 0 \text{ or } x > 0 \, , \text{ so } -\pi < \arg(x+ıy) < \pi \, ,$$

$$:= \mathrm{sign}(y)\cdot(1 - \mathrm{sign}(x))\pi/2 \quad \text{otherwise, where } \mathrm{sign}(\dots) := \pm 1 \text{ always.}$$

$$\exp(x + ıy) := e^x\cdot(\cos(y) + ı\cdot\sin(y)) \, .$$

$$\ln(z) := \ln(|z|) + ı\cdot\arg(z) \, . \quad \text{This principal value has } -\pi \le \mathrm{Im}(\ln(z)) \le \pi \, .$$

$$z^w := \exp(w\cdot\ln(z)) \quad \text{except } z^0 := 1 \text{ for } \textit{all} \; z \, , \text{ and } 0^w := 0 \text{ if } \mathrm{Re}(w) > 0 \, .$$

$$\sqrt{z} := z^{1/2} \, . \quad \text{This principal value has } \mathrm{Re}(\sqrt{z}) \ge 0 \, .$$

$$\mathrm{arctanh}(z) := (\ln(1+z) - \ln(1-z))/2 \qquad = -\mathrm{arctanh}(-z) \, .$$

$$\arctan(z) := \mathrm{arctanh}(ız)/ı \qquad = -\arctan(-z) \, .$$

$$\mathrm{arcsinh}(z) := \ln(z + \sqrt{(1 + z^2)}) \qquad = -\mathrm{arcsinh}(-z) \, .$$

$$\arcsin(z) := \mathrm{arcsinh}(ız)/ı \qquad = -\arcsin(-z) \, .$$

$$\mathrm{arccosh}(z) := 2\cdot\ln(\sqrt{((z+1)/2)} + \sqrt{((z-1)/2)}) \, .$$

$$\arccos(z) := 2\cdot\ln(\sqrt{((1+z)/2)} + ı\sqrt{((1-z)/2)})/ı \qquad = \pi/2 - \arcsin(z) \, .$$

Exercise 9: Locate the locus in the complex z-plane of each formula's discontinuities, if any.

These formulas' discontinuities, their *slits*, are located in the most commonly expected places. Also in accord with consensus are the values taken on the slits; acquiescence to the convention sign(0) := +1 is tantamount to attaching each slit to its side reached by going counter-clockwise around its one finite end. But this *Counter-Clockwise Continuity* is too simple a rule to work for a function whose slit is a finite line segment. Consequently some of the usual definitions of

arcsec(z) := arccos(1/z) , arccsc(z) := arcsin(1/z) , arccot(z) = arctan(1/z) ,
arcsech(z) := arccosh(1/z) , arccsch(z) := arcsinh(1/z) and arccoth(z) := arctanh(1/z)

may change one day as arccot did; it used to be $\mathrm{arccot}(z) := \pi/2 - \arctan(z)$ until about 1967, but now its slit is a finite line segment joining logarithmic branch-points at $z = \pm ı$ and poked at $z = 0$. The usual definition of arcsech(z) violates counter-clockwise continuity around $z = 0$.

These and many other annoying anomalies, like $\sqrt{(1/z)} \ne 1/\sqrt{z}$ and $\arg(\bar{z}) \ne -\arg(z)$ just when $z < 0$, go away when a signed zero is introduced, though it brings a new anomaly many people find more annoying, namely that $-4 + ı0 = -4 - ı0$ but $2ı = \sqrt{(-4 + ı0)} \ne \sqrt{(-4 - ı0)} = -2ı$. This is treated, along with other perplexing examples and numerically stable algorithms for the formulas above, in my paper "Branch Cuts for Complex Elementary Functions, or Much Ado About Nothing's Sign Bit", pp. 165-211 in *The State of the Art in Numerical Analysis* (1987) ed. by A. Iserles & M.J.D. Powell for the Clarendon (Oxford Univ.) Press.

Divided differences of transcendental functions cannot be simplified to eliminate the division …/(z–w) without incurring an integral; see Hermite's formulation above. Names have been given to some instances like $\exp^{\dagger}(\{z, -z\}) = \sinh(z)/z$ and $\ln^{\dagger}(\{1+z, 1-z\}) = \mathrm{arctanh}(z)/z$; and formulas like $\tan^{\dagger}(\{z, w\}) = (1 + \tan(w)\cdot\tan(z))\cdot\tan(z-w)/(z-w)$ often attenuate roundoff.

Exercise 10: Verify from the definitions of exp and ln above that they are complex analytic functions because their real and imaginary parts satisfy the Cauchy-Riemann equations. Find a short *algebraic* (not transcendental) expression for the derivative of ln and of each arc… .

Harmonic Conjugates

A complex function $f(z)$ of a complex argument z is called *Analytic* when it is complex differentiable on an open domain in the z-plane. We have seen that such an $f(z)$ decomposes into real and imaginary parts that must satisfy the Cauchy-Riemann equations on its domain:

$$f(x+\imath y) = g(x, y) + \imath h(x, y) , \quad \partial g(x, y)/\partial x = \partial h(x, y)/\partial y , \quad \partial g(x, y)/\partial y = -\partial h(x, y)/\partial x .$$

In other words, the Cauchy-Riemann equations are *necessary* for analyticity. They are *sufficient* too because, whenever they are satisfied by given functions g and h , these define a complex function $f := g + \imath h$ whose 2-vector interpretation's derivative $\mathbf{f'}$ is one of the special 2-by-2 matrices isomorphic with complex numbers; then the complex derivative is

$$f' = \partial g/\partial x + \imath\partial h/\partial x = \partial h/\partial y - \imath\partial g/\partial y .$$

What if g is given but not h ? Can we determine whether g is the real part of an analytic function f and, if so, then recover $f = g + \imath h$ from g ? Yes, and yes, to a degree.

Wherever g and h satisfy the Cauchy-Riemann equations in some open domain they are both *Harmonic Functions*: therein because each must satisfy *Laplace's Equation*:

$$\partial^2 g/\partial x^2 + \partial^2 g/\partial y^2 = 0 \quad \text{and} \quad \partial^2 h/\partial x^2 + \partial^2 h/\partial y^2 = 0 .$$

These equations follow from the Cauchy-Riemann equations and the observation that the order of differentiation can be reversed, $\partial(\partial g/\partial x)/\partial y = \partial(\partial g/\partial y)/\partial x$, provided the derivatives are all continuous. This proviso will be assumed here even though it could have been deduced instead. Consequently, only harmonic functions are eligible to be the real parts (or the imaginary parts) of complex analytic functions.

The imaginary part h of a complex analytic function $f = g + \imath h$ is called a *Harmonic Conjugate* (not *complex* conjugate) of the real part g . Both of them are harmonic in some open domain wherein they satisfy the Cauchy-Riemann equations. Then g is a harmonic conjugate of $-h$, not h . Either g or h determines the other minus an arbitrary real constant; either determines f minus an arbitrary real or imaginary constant, as we shall see next.

Books exhibit several ways to recover an analytic $f(x+\imath y) = g(x, y) + \imath h(x, y)$ from a given harmonic $g(x, y)$. Most textbooks do it this way:

Define $H(x, y) := \int \partial g/\partial x \, dy$, and then obtain $h(x, y) := H(x, y) - \int (\partial H/\partial x + \partial g/\partial y)dx - C$ for an arbitrary constant C . The claim is that $f := g + \imath h$ is analytic. To justify the claim we need merely verify that g and h satisfy the Cauchy-Riemann equations:

$$\begin{aligned}
\partial h/\partial x &= \partial H/\partial x - (\partial H/\partial x + \partial g/\partial y) = -\partial g/\partial y , \text{ as it should, and} \\
\partial h/\partial y &= \partial H/\partial y - \partial(\textstyle\int (\partial H/\partial x + \partial g/\partial y)dx)/\partial y \\
&= \partial H/\partial y - \textstyle\int (\partial(\partial H/\partial x)/\partial y + \partial^2 g/\partial y^2)dx \quad \text{if all derivatives are continuous} \\
&= \partial H/\partial y - \textstyle\int (\partial(\partial H/\partial y)/\partial x + \partial^2 g/\partial y^2)dx \\
&= \partial g/\partial x - \textstyle\int (\partial(\partial g/\partial x)/\partial x + \partial^2 g/\partial y^2)dx \\
&= \partial g/\partial x , \quad \text{as it should, because } \partial^2 g/\partial x^2 + \partial^2 g/\partial y^2 = 0 .
\end{aligned}$$

The integrals above have been written as *Indefinite Integrals* to hide an arbitrary constant that lurks within h . Another way to deal with that constant is to choose a point (x_o, y_o) inside the domain where g is harmonic and arbitrarily set $h(x_o, y_o) := 0$; this is tantamount to using *Definite Integrals*, first $H := \int\ldots dy$ running from (x, y_o) to (x, y) , and then $h := H - \int\ldots dx$

running the integration from (x_o, y) to (x, y) . But then these paths of integration must stay within the domain wherein g was given harmonic, thus restricting the recovery of h and $f = g+\imath h$ to whatever subregion of the domain is reachable by such paths from (x_o, y_o) . For example, if (x_o, y_o) is centered in a narrow rectangular domain whose edges make angles of $\pm\pi/4$ with the real and imaginary axes, the reachable subregion is a small hexagon; …

Exercise 11: Explain why.

Of course, (x_o, y_o) may be moved around the domain to reach other parts of it, but then h may change by some additive constant. Must all these changes be consistent with one function h over the whole domain ? Not necessarily**!**

Exercise 12: Except at the origin in the $(x + \imath y)$-plane, $g(x, y) := \ln(x^2 + y^2)$ is harmonic. Its harmonic conjugate can be recovered by using the recipe above; do so, and show that no choice of constants yields one harmonic conjugate h continuous throughout the whole domain of g .

Another way to recover h and f from a given harmonic g can be found in many textbooks; they use …

Green's Theorem in the Plane: $\int_{\partial R} (P\cdot dx + Q\cdot dy) = \iint_R (\partial Q/\partial x - \partial P/\partial y)\, dx\, dy$ wherein P(x, y) and Q(x, y) are continuously differentiable functions, and R is a plane region whose boundary ∂R is a piecewise smooth closed curve traversed during the first integration in a direction that puts the interior of R on the left.

Green's theorem is the flattened (into two dimensions) version of the three-dimensional …
Stokes' Theorem: $\int_{\partial R} \mathbf{v}\bullet d\mathbf{r} = \iint_R \text{curl}(\mathbf{v})\bullet\mathbf{n}\, dR$ wherein v is a continuously differentiable 3-vector-valued function of position in a 3-dimensional vector space, R in this space is a smooth surface whose edge is a piecewise smooth curve ∂R traversed for the first integral in infinitesimal steps d**r** , and **n** is the unit normal, at a point on R where dR is the infinitesimal element of area during the second integration, oriented according to the *Right-Hand Rule* viewed from the traversal of ∂R .

Green's theorem is often used to prove Stokes'. To obtain Green's from Stokes', choose for **v** a vector in the same plane as R with components P and Q therein, thereby ensuring that curl(**v**) be parallel to the plane's **n** .

Given any harmonic function g , a continuously differentiable solution of Laplace's equation on some open domain, set $P := -\partial g/\partial y$ and $Q := \partial g/\partial x$ into Green's theorem to infer that the integral

$$\int_{\partial R} (-\partial g/\partial y\cdot dx + \partial g/\partial x\cdot dy) = \iint_R (\partial^2 g/\partial x^2 + \partial^2 g/\partial y^2)\, dx\, dy = 0$$

around the boundary ∂R of every piecewise smoothly bounded subregion R inside the domain of g . Therefore we may select any finite point (x_o, y_o) inside that domain and define

$$h(x, y) := \int(-\partial g/\partial y\cdot dx + \partial g/\partial x\cdot dy)$$

integrated along any piecewise smooth path from (x_o, y_o) to (x, y) that stays strictly inside that domain. This h(x, y) is defined independent of the path *PROVIDED* every two such paths with the same endpoints (x_o, y_o) and (x, y) enclose between them only points interior to that domain wherein g satisfies Laplace's equation; the points between the paths constitute the subregion R for Green's theorem, which implies equality of the integrals along both paths.

We shall explore soon that proviso about points between the paths; first what does "between" mean ? A point lies between two paths just when every ray from that point to ∞ crosses the paths, taken together as one closed curve, an odd number of times counting the curve's self-crossings as multiple ray-crossings, two for an ✕ self-crossing, three for ✻, four for ✳, etc.

How do we confirm that this h , defined as a path-independent integral, is truly a harmonic conjugate of g ? We compute the partial derivatives of h . For this purpose we extend the path of integration along a line parallel to one of the coordinate axes. Parallel to the x-axis, $dy = 0$ and consequently $\partial h(x, y)/\partial x = \partial(\int(-\partial g/\partial y\cdot dx))/\partial x = -\partial g/\partial y$; similarly $\partial h/\partial y = \partial g/\partial x$, so g and h satisfy the Cauchy-Riemann equations as conjugates should.

Since our path-independent definition of h says $h(x, y) = \mathrm{Im}(\int(\partial g/\partial x - \imath\partial g/\partial y)(dx + \imath dy))$ it defines an analytic function $f(x + \imath y) := g(x, y) + \imath h(x, y)$ as a path-independent integral $f(z) := \int f'(z)dz = \int(\partial g/\partial x - \imath\partial g/\partial y)(dx + \imath dy)$ of its derivative $f'(x + \imath y) = \partial g/\partial x - \imath\partial g/\partial y$, to within an additive constant. In fact, every analytic function is the path-independent integral of its derivative so long as paths are restricted to the interior of the function's domain. (This is not the case for every differentiable real function of a real variable; some real derivatives oscillate too violently to be integrated.) Later we shall learn every analytic function's integral is path-independent too so long as paths and all points between them stay inside the function's domain.

Paths pose problems when at least one point not in the domain lies between them. At the cost of over-simplifying the subject, the problems can be dispelled by restricting attention to *Simply Connected* domains; in the plane these are domains without holes. Other characterizations of such domains include …

- Whenever two paths inside the domain have the same end-points,
  all points between the paths lie inside the domain too.
- Every closed curve inside the domain can be shrunk continuously, all the while
  remaining inside the domain, to a point inside the domain.

(The last characterization also characterizes simply connected domains of dimensions higher than 2 , and these can contain bubbles but not holes; for example, a cantaloupe's edible part is simply connected but a donut is not.) On any simply connected open domain in the plane, every function g harmonic on that domain has a harmonic conjugate h defined uniquely, but for an additive constant, everywhere on that domain by the foregoing path-independent integral.

The construction of a unique (but for an additive constant) harmonic conjugate h of g is the flattened version of the unGrad operation upon an irrotational flow in Euclidean 3-space:

If $\mathbf{q}(\mathbf{v})$ is a continuously differentiable 3-vector function of position $\mathbf{v}$ in a simply connected 3-dimensional domain whereon $\mathrm{curl}(\mathbf{q}) = \mathbf{o}$, then $\mathbf{q}(\mathbf{v}) = \mathrm{grad}(\text{Ø}(\mathbf{v}))$ for some scalar *Potential* function

$$\text{Ø}(\mathbf{v}) := \mathrm{unGrad}(\mathbf{q}) := \int \mathbf{q}(\mathbf{v})\bullet d\mathbf{v}$$ independent of the path of integration.

Exercise 13: How should $\mathbf{q}$ be determined by g to get Ø = h ?

Thus, by applying either Green's Theorem or the unGrad operator to the derivatives of a harmonic function g , we may recover its harmonic conjugate h uniquely, to within an additive constant, as a path-independent integral of the derivatives of g *provided* all points between paths lie inside the domain of g . The *proviso* is necessary for some harmonic functions, as in Exercise 12, but not all. …

Exercise 14: Prove that $g(x, y) := x/(x^2 + y^2)$ is a harmonic function whose conjugate is obtainable from a path-independent integral despite that its domain is not simply connected.

Exercise 15: On a simply connected domain consisting of the whole (x, y)-plane except for a slit cut along an arbitrarily chosen smooth *Simple* (not self-intersecting) curve joining 0 to ∞, the harmonic function $g(x, y) := \ln(x^2 + y^2)$ has a harmonic conjugate $h(x, y)$. Show that $\big(y - x\cdot\tan(h(x, y)/2)\big)/\big(x + y\cdot\tan(h(x, y)/2)\big)$ stays constant (perhaps ∞) throughout the domain while h runs through a range of real values whose extremes cannot differ by less than 4π, so no single-valued expression of the form 2 arctan(…) + constant can match $h(x, y)$.

Exercise 16: The *Critical Points* of $g(x, y)$ are the points (x, y) where $\partial g/\partial x = \partial g/\partial y = 0$; they may be maxima, minima or saddle-points of g. Show that conjugate harmonic functions have the same critical points. (Later we shall learn that none of these can be local maxima nor minima interior to the functions' domain.)

Exercise 17: The *Level Lines* of $g(x, y)$ are the curves in the (x, y)-plane on each of which g is constant. Show that the level lines of two harmonic conjugates form a family of *Orthogonal Trajectories*: one function's level lines intersect the other's orthogonally except at critical points. Orthogonality alone does not imply harmonic conjugacy; show that the level lines of $2x^2 + y^2$ and of y^2/x are orthogonal trajectories though neither function is harmonic, much less conjugate.

(However, if two harmonic functions' level lines form orthogonal trajectories, the functions can be proved to be each a constant multiple of the other's conjugate.)

Exercise 18: Suppose $g(x, y) = g(x, -y)$ is harmonic on a domain that includes a segment of the real (x-) axis in its interior; deduce that g has a harmonic conjugate $h(x, y) = -h(x, -y)$, and therefore $f(x + \imath y) := g(x, y) + \imath h(x, y)$ is an analytic function that is real on that segment of the real axis.

This justifies the term "Real Analytic Function" for any complex analytic function $f(z)$ that is real on a segment of the real z-axis strictly inside the domain of f, though some extra work is needed to deduce that this f must satisfy $f(x + \imath y) = \overline{f(x - \imath y)}$. The simplest way uses *Taylor Series* and *Analytic Continuation, q.v.* Every standard elementary function (or if multi-valued its *Principal Value*) discussed in the Chapter IV of our text, *Notes on Complex Function Theory* by Prof. Donald Sarason (1994), is a real analytic function.

Exercise 19: By integrating derivatives of a harmonic *function* $g(x, y)$ we recover one of its harmonic conjugates $h(x, y)$ and then set $f(x + \imath y) := g(x, y) + \imath h(x, y)$ to recover an analytic function $f(z)$. The recovery of an analytic *expression* f from a harmonic *expression* g that is also a real analytic function of each of its arguments can be achieved more easily by setting $f(z) := 2g((\bar{z}_o+z)/2, \imath(\bar{z}_o-z)/2)$ for any $z_o = x_o + \imath y_o$ inside the domain of g; explain why. If this procedure fails to recover a *function* f analytic throughout the domain of g, explain why.

Analytic functions are simpler than harmonic conjugates because every analytic function f is the path-independent integral $f(z) = f(c) + \int_c^z f'(w)dw$ of its derivative f' along every path inside the domain of analyticity regardless of whether all points between paths lie inside that domain, regardless of whether the domain is simply connected. We do insist that the domain of an analytic function be connected; otherwise perverse things could happen like …

Exercise 20: Prove that if $f' = 0$ throughout its domain, but this domain is not connected, then analytic function f stays constant in each connected component of its domain, though perhaps a different constant in a different connected component of that domain, regardless of whether the component is simply connected.

The Complex Plane *vs*. the Euclidean Plane

(What follows is supplementary to Math. 185.) If complex arithmetic notation seems to describe geometry in the Euclidean plane neatly, first impressions may be misleading. Complex multiplication plays an ambivalent rôle.

It seems natural to identify complex $z = x + \imath y$ with row vector $\mathbf{z} = [x, y]$, and $w = u + \imath v$ with $\mathbf{w} = [u, v]$. Then length $\|\mathbf{z}\| = |z|$; and $\overline{w}\cdot z = \mathbf{w}\bullet\mathbf{z} + \imath\cdot\mathbf{w}\times\mathbf{z}$ yields both scalar product $\mathbf{w}\bullet\mathbf{z} := u\cdot x + v\cdot y = |w|\cdot|z|\cdot\cos(\arg(z/w))$ and cross-product $\mathbf{w}\times\mathbf{z} := u\cdot y - v\cdot x = |w|\cdot|z|\cdot\sin(\arg(z/w))$; here $\arg(z/w)$ is the angle through which $\mathbf{w}$ must turn to align with $\mathbf{z}$. At the same time, $\overline{w}\cdot z$ is identified with a vector of length $|w|\cdot|z|$ making an angle $\arg(z/w)$ with the real axis. The imaginary unit is both a vector $\imath$ pointing up and an operator $\imath\cdot$ that turns vectors through $\pi/2$.

Ambiguity can be benign. Let $z(ß)$ be a smooth (twice differentiable) complex function of a real variable $ß$ with $dz/dß \neq 0$ (to preclude corners or cusps). As $ß$ varies z runs along a *Rectifiable* curve C, which means z covers a distance $\int_{z=p}^{q} |dz|$ when it runs from point p to q along C. The unit (length) tangent to C at z is $t(z) := dz/|dz|$, and there $n := \pm\imath\cdot t$ is a unit normal; $n \perp t$. A natural sign for n may be chosen at points z on C where $dt/dz \neq 0$; there $n = \rho\cdot dt/|dz|$ for a *radius of curvature* $\rho > 0$, which means that a circle of radius ρ centered at $z + \rho\cdot n$ is tangent to C at z and matches C's curvature $dt/|dz|$ there; n is the *inward pointing* normal there. (Do you see why $dt/|dz| \perp t$? How well the circle matches C ?) At *points of inflection* (where $dt/dz = 0$), $\rho = +\infty$ and the sign of n becomes arbitrary. So far, complex arithmetic imposes no impediment.

Now let $u(z)$ be a smooth real function of position $z = x+\imath y$ in the Euclidean plane. No complex number " $u'(z)$ " can serve as derivative because real $du(z) \neq u'(z)\cdot dz$. Instead we define $\nabla u(z) := \partial u(x+\imath y)/\partial x + \imath\partial u(x+\imath y)/\partial y$ to be u's complex *Gradient* so that $du(z) = \mathrm{Re}(\, \nabla u(z)\cdot\overline{dz}\,)$. Here "Re" and conjugation "$\overline{dz}$" are the nuisances inflicted by mixing complex arithmetic with Euclidean vectors.

Exercise 21: Confirm that $\nabla u(z)$ is *normal* ($\perp$) to u's level line through z, that $\nabla u(z)$ points in the direction of infinitesimal motions dz that maximize $du/|dz|$, and that this maximum is $|\nabla u(z)|$. Use Green's theorem (p. 12 above) to prove $\int_{\partial R} \overline{\nabla u}\cdot dz = \imath\cdot\iint_R |\nabla|^2 u\cdot dx\cdot dy$ where the *Laplacian* $|\nabla|^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ involves 2nd derivatives assumed continuous in an open region R. Generally $\mathrm{Im}(\int \overline{\nabla u}\cdot dz)$ depends on the path of integration.

Next let $u(z)$ and $v(z)$ be smooth real functions of position $z = x+\imath y$, but not conjugate harmonic functions of (x, y), so that $w(z) := u(z) + \imath v(z)$ maps the Euclidean plane into itself but not conformally. Neither " $w'(z)$ " nor " $\nabla w(z)$ " provides a complex number to serve as the derivative of w which, like the 2-by-2 matrix $\boldsymbol{f'}$ on p. 1, has not two but four real elements. Still, complex multiplication of ∇ and $\overline{w}$ yields an interesting object $\nabla\overline{w} = (\partial/\partial x + \imath\partial/\partial y)(u - \imath v) = \nabla\bullet\mathbf{w} - \imath\cdot\nabla\times\mathbf{w}$ combining the *Divergence* $\nabla\bullet\mathbf{w} := \partial u/\partial x + \partial v/\partial y$ with the *Scalar Curl* $\nabla\times\mathbf{w} := \partial v/\partial x - \partial u/\partial y$. Do you see why $\nabla\overline{w}$ vanishes when $\overline{w}(z)$ is an analytic function of z ? Otherwise $\nabla\overline{w}$ accounts for the path-dependence of $\int_C \overline{w}(z)\cdot dz$, which need not vanish if C is a loop, as follows: …

Exercise 22: Confirm a complex analog $\int_{\partial R} \overline{w}(z)\cdot dz = \imath\cdot\iint_R \nabla\overline{w}\cdot dx\cdot dy$ of Green's theorem by applying it twice. Verify that $\overline{w}\cdot dz = \mathbf{w}\bullet d\mathbf{z} + \imath\cdot\mathbf{w}\times d\mathbf{z} = (\mathbf{w}\bullet\mathbf{t} + \imath\cdot\mathbf{w}\bullet\mathbf{n})\cdot|dz|$ where t is the unit tangent to ∂R so directed that R lies on the left, and $n := -\imath\cdot t$ is the *outward* (right-) pointing normal to ∂R. Then the complex Green's theorem yields Stokes' theorem $\int_{\partial R} \mathbf{w}\bullet\mathbf{t}\cdot|dz| = \iint_R \nabla\times\mathbf{w}\cdot dx\cdot dy$ and Gauss' Divergence theorem $\int_{\partial R} \mathbf{w}\bullet\mathbf{n}\cdot|dz| = \iint_R \nabla\bullet\mathbf{w}\cdot dx\cdot dy$ in the plane, thus condensing two renowned theorems into one cryptic equation $\int_{\partial R} \overline{w}(z)\cdot dz = \imath\cdot\iint_R \nabla\overline{w}\cdot dx\cdot dy$.

Into the last equation substitute $w := g\cdot\nabla h$, where $g(z)$ and $h(z)$ are smooth real functions of position $z = x+\imath y$, to get $\int_{\partial R} g\cdot\overline{\nabla h}\cdot dz = \imath\cdot\iint_R (\, \nabla g\cdot\overline{\nabla h} + g\cdot|\nabla|^2 h\,)\cdot dx\cdot dy$. This will figure in the characterization of harmonic functions as solutions of the following variational problem: Suppose region R is inside the domain of a harmonic function h (so $|\nabla|^2 h = 0$ in R), and suppose at least part of the boundary ∂R does not cut level-lines of h orthogonally; along ∂R's remainder, if any, h's *normal derivative* " $\partial h/\partial\mathbf{n}$ " $:= \mathrm{Re}(\overline{\nabla h}\cdot n) = \mathrm{Im}(\overline{\nabla h}\cdot t) = \mathrm{Im}(\overline{\nabla h}\cdot dz)/|dz|$ vanishes. Except on that remainder, suppose $g = 0$ on ∂R. Thus, $g+h$ runs over smooth functions whose boundary values on ∂R match h there ("Dirichlet conditions") except perhaps on that remainder where $\partial h/\partial\mathbf{n} = 0$ ("Neumann conditions"). Of all such smooth functions, the one that minimizes $\iint_R |\nabla(g+h)|^2 dx\cdot dy$ turns out to be h because
$\iint_R |\nabla(g+h)|^2 dx\cdot dy = \iint_R (|\nabla g|^2 + |\nabla h|^2 + 2\mathrm{Re}(\nabla g\cdot\overline{\nabla h}))\cdot dx\cdot dy = \iint_R (|\nabla g|^2 + |\nabla h|^2 - 2g\cdot|\nabla|^2 h)\cdot dx\cdot dy + \mathrm{Im}(\int_{\partial R} g\cdot\overline{\nabla h}\cdot dz)$
$= \iint_R (|\nabla g|^2 + |\nabla h|^2)\cdot dx\cdot dy \geq \iint_R |\nabla h|^2 dx\cdot dy$, with equality just when $g = 0$ in R. In his thesis G. Riemann took for granted that a minimizing harmonic h must exist for any given piecewise smooth boundary values on ∂R.

Summary of the Next Few Topics:

1: Every analytic function f is the path-independent integral $f(z) = f(c) + \int_c^z f'(w)dw$ of its derivative f' along every path inside the domain of analyticity regardless of whether all points between paths lie inside that domain, regardless of whether the domain is simply connected. Proof: Green's Theorem or unGrad exploit the Cauchy-Riemann equations; and since f and the integral of its derivative have the same derivative throughout the domain, they must differ by a constant thereon.

2: Every continuous function f whose integral $F(z) := \int_c^z f(w)dw$ is path-independent on some open domain (connected to the point c) is the derivative $f(z) = F'(z)$ of its integral, which is therefore analytic on that domain. Proof: $F^{\dagger}(\{z+\Delta z,z\}) - f(z) \to 0$ as $|\Delta z| \to 0$.

3: The integral $\int f(z)dz$ of every analytic function f is path-independent and therefore analytic on every *simply-connected* open subset of the domain of f . (Cauchy-Goursat theorem)

4: Cauchy's Integral Formula: $f^{[n]}(z) = n! \int_C f(w)(w-z)^{-1-n}dw/(2\pi\imath)$ for every integer $n \ge 0$ if z is inside a simple closed curve C inside which f is analytic, on and near which f is piecewise continuous; so all derivatives $f^{[n]}(z)$ exist and are analytic too. (Goursat's proof)

5: Every continuous function, whose integral is path-independent throughout some domain (it is evidently connected, but perhaps not simply), is analytic thereon. (Morera's theorem)

6: Wherever f is analytic, so is $f^{\dagger}$, since $f^{\dagger} \to f'$. This is an example of a *Removable Singularity*: If inside an open domain F is analytic everywhere except perhaps at one interior point around which F is bounded, then F can be (re)defined at that point to render it analytic there too. (Riemann's removal of a singularity)

7: The only bounded entire functions are constants. (Liouville's theorem)

8: Every complex polynomial has as many zeros as its degree, counting multiplicities. (Gauss)

9: Every analytic function equals its average value on a concentric circle. (Gauss)

10: The *Taylor Series* of an analytic $f(z) = \sum_{n\ge0} (z-z_o)^n f^{[n]}(z_o)/n!$ converges absolutely and is term-by-term differentiable and integrable within its circle of convergence, on which lies the singularity of f nearest z_o . The series diverges outside this circle. At any point on this circle where the series converges it converges to the non-tangential limit of f . (Abel's theorem)

11: Every analytic function's magnitude takes its maximum value over its domain somewhere on its boundary. Every non-constant analytic functions maps interior points of its domain only to interior points of its range. (Maximum Modulus Theorem = Open Mapping Theorem)

12: Every analytic function's magnitude takes its minimum value over its domain somewhere on its boundary if not at its zero(s) inside the domain. (D'Alembert's principle)

13: If f and g are analytic throughout the same domain, and if $|f - g| < |g|$ on its boundary, then f and g have the same number of zeros inside that domain. (Rouché's Theorem)