Exercise 2 p. 43: Let â be a complex number of unit modulus and ç an irrational real number. Prove that the values of $â^ç$ form a dense subset of the unit circle.

Solution: Say â = exp(ıµ) for some real µ ; then the values of $â^ç$ in question are the values of exp(ı·(µ + 2nπ)·ç) as n runs through all integers. These are the same values as are taken by exp(ı·(µ·ç + 2(n·ç – k)π)) as k runs through integers like [n·ç], the biggest integer no bigger than n·ç . Let f(x) := x – [x] denote the *fractional part* of a real number x , so that 0 ≤ f(x) < 1 and f(x + y) = f(f(x) + f(y)) . To solve the problem we must explain why the values taken by f(n·ç) as n runs through all integers are dense in the interval 0 < f(n·ç) < 1.

The proof goes by contradiction. Since every member of the sequence $\{f(n·ç)\}_{n = 1,2,3,…}$ lies strictly between 0 and 1 (else ç would be rational), nothing is lost by first joining the ends of the interval (0, 1) to wrap it around a circle, and then treating the sequence's members as points on that circle. Suppose for argument's sake that the sequence were not dense. Then some open arc **A** on the circle would be *empty* (contain no points of the sequence). Let **Â** be the widest empty arc containing **A** ; such a maximal arc **Â** would have to exist because its endpoints would be limits of monotonic sequences of endpoints of ever wider empty arcs. For each m = 0, 1, 2, 3, … the arc f(**Â** – m·ç) := { all points f(α – m·ç) generated by letting α run through **Â** } would have to be empty too since f(n·ç) could fall in f(**Â**–m·ç) only if f((n+m)·ç) fell in f(**Â**) = **Â** . All the arcs f(**Â** – m·ç) would be different because the equation f(α – m·ç) = f(α – k·ç), if true for different integers m and k , would imply that ç is rational. All the arcs f(**Â** – m·ç) would be disjoint (non-overlapping) too because the overlap of f(**Â** – m·ç) with f(**Â** – k·ç) for m > k would imply the overlap of f(**Â** – (m–k)ç) with f(**Â**) = **Â** , contradicting the maximality of **Â** . But no circle can hold infinitely many disjoint open arcs f(**Â** – m·ç) of equal nonzero widths, so they cannot be empty open arcs after all; the sequence $\{f(n·ç)\}_{n = 1,2,3,…}$ is dense as claimed.

I believe this proof can be traced back to L. Kronecker in the second half of the 19th century.

Satvik Beri's Solution:
First this problem will be reduced to a search for integers m and n that make | r + ç·n – m | arbitrarily small after irrational ç and any arbitrary real r have been fixed. After this the two sought integers will be proved to exist.

What values are taken by $â^ç$? Since |â| = 1 we can write â = $e^{ı(µ + 2n·π)}$ for some fixed real µ between ±π and any integer n , and then by definition the set of values of $â^ç$ consists of the values taken by exp(ı·ç·(µ + 2n·π)) as n runs through all integers. Since all of these values have magnitude 1 they all lie on the unit circle.

Any point on the unit circle can be expressed as $e^{ıß}$ for some real ß between ±π inclusive. To prove that the values of exp(ı·ç·(µ + 2n·π)) are dense in the unit circle we have to show that, given any such ß , there are integers n that make differences | exp(ı·ç·(µ + 2n·π)) – $e^{ıß}$ | arbitrarily small. This can be done, because exp(ı·(…)) is a continuous periodic function of a real argument, by finding integers n that make | (ç·(µ + 2n·π) – ß) **mod** 2π | arbitrarily small, which is tantamount to searching, after ç and µ have been fixed, for integers n and m that make | (ç·(µ + 2n·π) – ß) – 2m·π | arbitrarily small for each given real ß.

Divide out 2π and set $r := (ç\cdot\mu - ß)/(2\pi)$; now we search for integers n and m that make $|r + ç\cdot n - m|$ arbitrarily small after real r and irrational ç have been fixed. What "arbitrarily small" means is that, having chosen any big integer $K >> 2$, we can find integers m and n that make $|r + ç\cdot n - m| < 1/(2K)$. What follows will prove that such integers m and n exist.

Let $f(x)$ be the fractional part of any real x as described above, and break the open interval $0 < x < 1$ into K separated fragments $0 < x < 1/K$, $1/K < x < 2/K$, $2/K < x < 3/K$, …, and $1 - 1/K < x < 1$. As k runs through 1, 2, 3, …, K the K values $f(k\cdot ç)$ scatter into the insides of those fragments because no value $f(k\cdot ç)$ can be rational when ç is irrational. At least one value $f(\pm k\cdot ç)$ must fall into the first fragment, $0 < f(\pm k\cdot ç) < 1/K$; otherwise some two of the K values $f(k\cdot ç)$, say $f(k_1\cdot ç)$ and $f(k_2\cdot ç)$ for $0 < k_1 < k_2 \le K$, would have to fall into the same one of the other K–1 fragments, making $|f(k_2\cdot ç) - f(k_1\cdot ç)| < 1/K$ and implying either $0 < f(k_2\cdot ç) - f(k_1\cdot ç) = f((k_2 - k_1)\cdot ç) < 1/K$ or else $0 < f(k_1\cdot ç) - f(k_2\cdot ç) = f((k_1 - k_2)\cdot ç) < 1/K$. Either way, $0 < k := k_2 - k_1 < K$ or $0 > k := k_1 - k_2 > -K$ has $0 < f(k\cdot ç) < 1/K$ as claimed.

Having chosen this k , let j be the integer, positive or negative, nearest $-r/f(k\cdot ç)$, so that $|j + r/f(k\cdot ç)| \le 1/2$. Then set integers $n := j\cdot k$ and $m := j\cdot[k\cdot ç] = j\cdot(k\cdot ç) - j\cdot f(k\cdot ç)$ to get $|r + ç\cdot n - m| = |r + j\cdot k\cdot ç - j\cdot k\cdot ç + j\cdot f(k\cdot ç)| = f(k\cdot ç)|j + r/f(k\cdot ç)| \le f(k\cdot ç)/2 < 1/(2K)$. End.