Straight Lines

The straight line £ through distinct points b and b+ç is the locus of points z satisfying $\mathrm{Im}((z–b)/ç) = 0$ or equivalently $\mathrm{Im}((z–b)\cdot\overline{ç}) = 0$, wherein $\overline{ç}$ is the complex conjugate of ç. The distance from other points z to £ is $|z–£| := |\mathrm{Im}((z–b)\cdot\overline{ç})|/|ç|$ because the point on £ nearest z is $z – ı\cdot\mathrm{Im}((z–b)\cdot\overline{ç})/\overline{ç} = z – ıç\cdot\mathrm{Im}((z–b)/ç)$.

Where do two straight lines intersect? Unless they are parallel, they intersect in just one point. Let us find it for lines whose equations are $\mathrm{Im}(\overline{ç}\cdot(z–b)) = 0$ and $\mathrm{Im}(\overline{Ç}\cdot(z–B)) = 0$. Put these equations into the form $\overline{ç}\cdot(z–b) = ç\cdot\overline{(z–b)}$ and $\overline{Ç}\cdot(z–B) = Ç\cdot\overline{(z–B)}$, and then eliminate $\overline{z}$ to get the intersection point $z = (\, Ç\cdot\mathrm{Im}(\overline{ç}\cdot b) – ç\cdot\mathrm{Im}(\overline{Ç}\cdot B)\,)/\mathrm{Im}(Ç\cdot\overline{ç})$.

Parabolas

The *Parabola* whose directrix is £ and whose focus is at f not on £ is the locus of points z equidistant from f and from £ ; these points z satisfy the √-free equation $|z–f|^2 = |z–£|^2$,

i.e. $|z–f|^2 = \mathrm{Im}((z–b)/ç)^2\cdot|ç|^2$, or $|z–f|^2 = \mathrm{Im}((z–b)\cdot\overline{ç})^2/|ç|^2$.

At first sight this equation factors into two equations $|z–f| = \pm|z–£|$, but no value of z can satisfy $|z–f| = –|z–£|$ so long as f does not lie on £ . The parabola lies entirely on the same side of £ with f since $|z–f| > |z–£|$ when z and f lie on opposite sides of £ . Therefore, when z lies on the parabola, $\mathrm{Im}((z–b)\cdot\overline{ç})$ and $\mathrm{Im}((f–b)\cdot\overline{ç})$ must have the same nonzero sign, and so $|z–f| = \mathrm{sign}(\mathrm{Im}((f–b)\cdot\overline{ç}))\cdot\mathrm{Im}((z–b)\cdot\overline{ç})/|ç|$.

We resume now the discussion of solutions to problems issued on 28 Aug. 2006. They are …

Problem 6: Let distinct points $\pm f$ be the two *Foci* of a *Central Conic Section* drawn in the complex z-plane. This curve is classified according to the equation satisfied by a real constant Ç and complex variable z on the locus:

Ellipse: $|z–f| + |z+f| = 2Ç|f| \ge 2|f|$.

Hyperbola: $–2|f| \le |z–f| – |z+f| = \pm 2Ç|f| \le 2|f|$.

(a) Show how the inequalities descend from the *Triangle Inequality* $|z+w| \le |z|+|w|$.

(b) Show why z , if it satisfies *one* of the foregoing equations, must satisfy also

$$|z|^2 + |f|^2 – (\mathrm{Re}(z/f)|f|/Ç)^2 – Ç^2|f|^2 = 0 .$$

(This equation is free of the square root implicit in |…| .)

(c) For which values Ç do these conics degenerate to straight lines or line segments?

(d) What happens to the ellipse if $f \longrightarrow 0$ and $Ç \longrightarrow +\infty$ while $|Çf|$ stays constant?

(e) Show that the hyperbola's *Asymptote*s' equation is $|z/f|^2 – (\mathrm{Re}(z/f)/Ç)^2 = 0$.

Problem 7: A conformal map $z \longleftrightarrow w$ is defined by two nonzero constants f and ß thus:

$$z = f(w/ß + ß/w)/2 , \quad \text{or equivalently} \quad w/ß = z/f \pm \sqrt{(z/f)^2 – 1} .$$

(a) Show that $z \pm f = f(w \pm ß)^2/(2ßw)$, and thus express the conformal map more symmetrically via the equation $(z + f)/(z – f) = ((w + ß)/(w – ß))^2$.

(b) Locate the map's *Poles*, points in one plane mapped to ∞ in the other.

(c) The map is mostly 2-to-1 in that it takes $w = w_1$ and $w = ß^2/w_1$ to the same z .
Locate the map's *Critical Points*, where the map is 1-to-1 instead of 2-to-1 .

(d) Show that pairs of concentric circles in the w-plane map to ellipses in the z-plane thus: For any real constant $\Omega > 0$ both circles $|w| = |ß|\Omega$ and $|w| = |ß|/\Omega$ map to the ellipse $|z+f| + |z-f| = (\Omega + 1/\Omega)|f|$, except for a degenerate case in which the circle $|w| = |ß|$ maps twice to the line segment joining foci $\pm f$.

(e) Show that pairs of straight lines through the origin in the w-plane map to hyperbolas with foci $\pm f$ in the z-plane thus: For any fixed angle Ø , the straight line through 0 in the w-plane traced by $w = \Omega \cdot ß \cdot \exp(\imath Ø)$ as Ω runs through all real values maps onto the hyperbola $|z+f - |z-f| = \pm 2|f|\cdot\cos(Ø)$; replacing Ø by –Ø yields another line mapped upon the same hyperbola except for two degenerate cases (which ?) of lines mapped to lines or line segments.

(f) At what angle(s) do the foregoing ellipses and hyperbolas intersect?

Central Conics: High-School Analytic Geometry in the Complex Plane

Let $\pm f$ be at the foci of a central conic section drawn in the complex z-plane. This curve is classified according to the equation satisfied by a real constant μ and complex variable z on the locus:

Ellipse: $|z-f| + |z+f| = 2\mu|f| \ge 2|f|$, so $\mu \ge 1$ here.
Hyperbola: $-2|f| \le |z-f| - |z+f| = \pm 2\mu|f| \le 2|f|$, so $-1 \le \mu \le 1$ here.

The inequalities come from the *Triangle Inequality* $|t \pm w| \le |t| + |w|$; for example try $t := z+f$ and $w := z-f$, or $t := 2f$, *etc*.

The ± sign in the hyperbola distinguishes its two branches. On the hyperbola's asymptotes, z satisfies $\mathrm{Re}(z/f) = \pm\mu|z/f|$ or, equivalently, $\mathrm{Im}(z\cdot e^{\pm\imath \arccos(\mu)}/f) = 0$ where $\imath^2 = -1$. For both conics the ends of the major axis are at $\pm\mu f$. The ends of the ellipse's minor axis are at $\pm\imath\sqrt{(\mu^2-1)}f$. (To translate the center of the conic from 0 to b merely replace z by z–b in all equations above, and add b to the foci and to the axes' ends.)

There are some degenerate and limiting cases:

Ellipse	with $f = 0 < \mu|f| = r$ is a circle of radius r. ($\mu = +\infty$.)
	with $\mu = 1$ is the line segment from f to $-f$.
Hyperbola	with $f = 0$ is the whole plane;
	with $\mu = 0 \ne f$ is the right bisector of line segment f to $-f$;
	with $\pm\mu = -1$ is a half-line emanating from $\pm f$.

To rid the conic's equation of the square roots implied by |…| we multiply together all the equation's four distinct conjugates, namely

“ $2\mu|f| + |z-f| + |z+f| = 0$ ” ,
“ $2\mu|f| - |z-f| - |z+f| = 0$ ” ,
“ $2\mu|f| - |z-f| + |z+f| = 0$ ” and
“ $2\mu|f| + |z-f| - |z+f| = 0$ ” ,

(only one of which can be satisfied in non-degenerate cases) to get one equation that, after division by $-16|f|^2$, simplifies to

$$\mu^2(|z|^2 + |f|^2) - (|f|\cdot\mathrm{Re}(z/f))^2 - \mu^4|f|^2 = 0 .$$

This one equation is the equation of …

- if $\mu^2 > 1$ then an ellipse ;
- if $\mu^2 = 1$ then the straight line through $\pm f$, twice;
- if $0 < \mu^2 < 1$ then both branches of an hyperbola;
- if $\mu = 0$ then the straight line right-bisector of the line segment joining $\pm f$, twice.

To get the equation of a circle of radius $r > 0$ centered at 0, divide the equation by μ^2 and let $f \longrightarrow 0$ and $\mu \longrightarrow +\infty$ while keeping $\mu\cdot|f| = r > 0$.

The hyperbola's asymptotes' equation is $|z/f|^2 - (\mathrm{Re}(z/f)/\mu)^2 = 0$, which factors into the two equations $|z/f| = \pm\mathrm{Re}(z/f)/\mu$ of two V-shaped figures consisting of half-lines emanating from 0 making angles of $\pm\arccos(\mu)$ with the two line segments from 0 to $\pm f$.

A Conformal Map of Lines onto Parabolas

For any complex constant f a 1-to-2 conformal map $z \longleftrightarrow w$ is defined either by $z = f + w^2$ or, equivalently, by $w = \pm\sqrt{z-f}$. The map is 1-to-1 at its two Critical Points $z = f$, $w = 0$, and $z = \infty$, $w = \infty$. Every distinct pair $\pm\Pi$ of parallel straight lines in the w-plane maps to a parabola P twice in the z-plane thus:

For any complex constants B and $Ç \neq 0$ the locus of $w := B + Ç\cdot\mu$ as μ runs through all real values is a straight line Π whose equation is $\mathrm{Im}((w-B)\cdot\overline{Ç}) = 0$. Parallel to Π in the w-plane is $-\Pi$, the locus of $w := -(B + Ç\cdot\mu)$ whose equation is $\mathrm{Im}((w+B)\cdot\overline{Ç}) = 0$. Both $\pm\Pi$ are mapped to the z-plane by $z = f + w^2$ onto a parabola P whose equation is

$$|z-f| = \left(\mathrm{Re}((z-f)/Ç^2) + 2\mathrm{Im}(B/Ç)^2 \right)\cdot|Ç|^2 .$$

f is the focus of P; and its directrix is the line, through $f - 2(Ç\cdot\mathrm{Im}(B/Ç))^2$ parallel to $\imath Ç^2$, whose equation is $\mathrm{Re}((z-f)/Ç^2 + 2\mathrm{Im}(B/Ç)^2) = 0$. P degenerates into a half-line emanating from f when $\mathrm{Im}(B/Ç) = 0$.

A Conformal Map of Circles and Lines onto Central Conic Sections:

A conformal map $z \longleftrightarrow w$ is defined by two nonzero complex constants f and ß thus:

$$z = (w/ß + ß/w)f/2 , \quad \text{or} \quad (z + f)/(z - f) = ((w + ß)/(w - ß))^2 , \quad \text{or} \quad w/ß = z/f \pm \sqrt{((z/f)^2 - 1)} .$$

Two points w and $ß^2/w$ in the w-plane map to/from the same point z in the z-plane except for the two Critical Points $z = \pm f$, $w = \pmß$, where $dz/dw = 0$. Poles are at $z = \infty$ when $w = 0$ or ∞; but because the map's multiplicity does not change in the poles' neighborhoods, the poles are not critical points.

Pairs of concentric circles centered at 0 in the w-plane map to ellipses twice in the z-plane with foci at $\pm f$ as follows:

For any constant $\Omega > 0$ both circles $|w| = |ß|\Omega$ and $|w| = |ß|/\Omega$ map onto the ellipse whose equation is $|z+f| + |z-f| = (\Omega + 1/\Omega)|f|$. The circle with $\Omega = 1$ maps twice to the line segment joining the foci $\pm f$. The two annuli between the outer circles and the inner one map to the interior of the ellipse traced twice. To see why this is so observe that in general

$$z \pm f = f(w/ß \pm 2 + ß/w)/2 = f(w \pm ß)^2/(2ßw) ,$$

so when w is on the circle $|w| = |ß|/\Omega$ for any constant $\Omega > 0$

$$\begin{aligned} |z+f| + |z-f| &= \Omega|f/ß^2|\cdot(|w + ß|^2 + |w - ß|^2)/2 \\ &= \Omega|f/ß^2|\cdot(2|w|^2 + 2|ß|^2)/2 \\ &= \Omega|f/ß^2|\cdot(|ß|^2/\Omega^2 + |ß|^2) = (\Omega + 1/\Omega)|f| \end{aligned}$$

as claimed. Replacing Ω by $1/\Omega$ in the w-plane changes nothing in the z-plane.

The circle $|w| = |ß|\Omega$ can be generated by setting $w := |ß|\Omega\cdot\exp(\imath Ø)$ and letting angle Ø run through all real values between $-\pi$ and $+\pi$. An alternative that requires no transcendental function is to set $w := |ß|\Omega\cdot(\mu + \imath)/(\mu - \imath)$ and let $\mu = \cot(Ø/2)$ run through all real values.

Pairs of straight lines through the origin in the w-plane map to hyperbolas twice in the z-plane with foci at $\pm f$ as follows:

For any fixed angle Ø a straight line through 0 in the w–plane is traced by $w = \Omega\cdot ß\cdot\exp(\imath Ø)$ as Ω runs through all real values; here $\exp(\imath Ø) = \cos(Ø) + \imath\cdot\sin(Ø)$ stays constant on the line. Two lines are mapped onto each hyperbola $|z+f| - |z-f| = \pm 2|f|\cdot\cos(Ø)$. To see why, recall that in general

$$z \pm f = f(w/ß \pm 2 + ß/w)/2 = f(w \pm ß)^2/(2ßw) ,$$

so on the line $w = \Omega\cdot ß\cdot\exp(\imath Ø)$ we find that

$$\begin{aligned} |z+f| - |z-f| &= |f|\cdot(|w+ß|^2 - |w-ß|^2)/|2ßw| \\ &= 2|f|\cdot \mathrm{Re}(\bar{ß}\cdot w)/|ßw| = 2|f|\cdot \mathrm{sign}(\Omega)\cdot\cos(Ø) . \end{aligned}$$

Replacing Ø by –Ø generates the same hyperbola in the z-plane from two lines in the w-plane that are distinct except in the degenerate cases $Ø = 0$ or $\pm\pi$, and $Ø = \pm\pi/2$.

Because the straight lines through 0 intersect circles centered at 0 orthogonally in the w-plane, and because the conformal map preserves angles of intersection except at its critical points, the hyperbolas and ellipses described above intersect orthogonally in the z-plane too.

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Where do two different conics intersect? They can intersect in at most four points. Finding these generally entails solving a quartic or cubic equation that can’t be solved using only finitely many real operations each restricted to one of $\{ +, -, \cdot, /, \sqrt[n]{\ } \}$. But special cases exist like …

Centers of the Circles of Apollonius of Perga (ca. 200 B.C.): A story for another day.

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