When are three distinct points *Collinear* ? When are four distinct points *Concyclic* ?

Here "collinear" means "on the same (real) straight line"; and "concyclic" means "on the same circle" including maybe a straight line regarded as a circle with infinite radius. The following problems elaborate problems #26 and #27 at the end of §1.2 of *Basic Complex Analysis* 3rd ed. by J.E. Marsden & M.J. Hoffman (1999).

0: Solve the equation $(\mathbf{w}-\mathbf{t})/(\mathbf{z}-\mathbf{t}) = (\mathbf{z}-\mathbf{t})/(\mathbf{z}-\mathbf{w})$ for one of the three complex numbers $\mathbf{t}$, $\mathbf{w}$ and $\mathbf{z}$ in terms of the others to show that, as points in the complex plane, they are situated at the vertices of an equilateral triangle. (Note that the equation is invariant under shifts of origin.)

Solution: Shifting the origin to $\mathbf{t}$ by subtracting $\mathbf{t}$ from every variable amounts to setting $\mathbf{t} := 0$ in the equation, turning it into $\mathbf{w}/\mathbf{z} = \mathbf{z}/(\mathbf{z}-\mathbf{w})$ whence $\mathbf{z}^2 - \mathbf{w}\cdot\mathbf{z} + \mathbf{w}^2 = 0$. Therefore $\mathbf{z} = \mathbf{q}\cdot\mathbf{w}$ for $\mathbf{q} := \frac{1}{2}(1 \pm \imath\sqrt{3}) = e^{\pm\imath\pi/3}$ (choose either value). Multiplication by $\mathbf{q}$ performs a rotation through $\pi/3 = 60^\circ$. $\mathbf{w}-\mathbf{z} = \bar{\mathbf{q}}\cdot\mathbf{w}$ is rotated the same amount in the opposite direction. And since $|\bar{\mathbf{q}}| = |\mathbf{q}| = 1$ we find that $|\mathbf{w}-\mathbf{z}| = |\mathbf{w}| = |\mathbf{z}|$, so the points $\mathbf{z}$, $\mathbf{w}$ and $\mathbf{t} = 0$ are the vertices of an equilateral triangle, as claimed.

1: Given three distinct *vectors* $\mathbf{t}$, $\mathbf{w}$ and $\mathbf{z}$ interpreted as displacements from some origin $\mathbf{o}$ to three distinct points in a real linear space of arbitrary dimension, what condition do these vectors satisfy just when the points are collinear? Why?

Solution: The three points are collinear just when $\mathbf{w}-\mathbf{t}$ and $\mathbf{z}-\mathbf{t}$ are *Linearly Dependent* (either parallel or anti-parallel in this case), which means $\mathbf{w}-\mathbf{t} = \mu\cdot(\mathbf{z}-\mathbf{t})$ for some real scalar $\mu \neq 0$. Replacing the equation's differences by any other two distinct differences among the three vectors merely changes the scalar. If the three vectors are actually complex numbers then the equation says that any difference quotient, like $(\mathbf{w}-\mathbf{t})/(\mathbf{z}-\mathbf{t})$, is real. This one is negative just when $\mathbf{t}$ lies between $\mathbf{w}$ and $\mathbf{z}$ on the line.

2: Given only the three *Euclidean distances* $\|\mathbf{t}-\mathbf{w}\|$, $\|\mathbf{w}-\mathbf{z}\|$ and $\|\mathbf{z}-\mathbf{t}\|$ between those three distinct points, what condition do the distances satisfy just when the points are collinear? Why?

Solution: The distances satisfy the *Triangle Inequality*; it says that the biggest of the three must equal or exceed the sum of the other two. Equality occurs if and only if the points are collinear and appropriately ordered: $\|\mathbf{z}-\mathbf{t}\| = \|\mathbf{w}-\mathbf{z}\| + \|\mathbf{t}-\mathbf{w}\|$ just when $\mathbf{w}$ lies between $\mathbf{t}$ and $\mathbf{z}$ on the line. Were distance not Euclidean, equality would be necessary but not necessarily sufficient for collinearity. (See <www.cs.berkeley.edu/~wkahan/MathH110/NORMlite.pdf>.)

3: Given four distinct *vectors* $\mathbf{q}$, $\mathbf{t}$, $\mathbf{w}$ and $\mathbf{z}$ interpreted as displacements from some origin $\mathbf{o}$ to four distinct points in Euclidean 3-space, what conditions do these vectors satisfy just when the points are concyclic? Why? (<www.cs.berkeley.edu/~wkahan/MathH110/Cross.pdf> may help.)

Solution: Temporarily let's assume no three of the four given points are collinear, and simplify the algebra by translating the origin $\mathbf{o}$ to $\mathbf{q}$, say, by subtracting $\mathbf{q}$ from every given vector.

Through any three non-collinear points $\mathbf{p}-\mathbf{v}$, $\mathbf{p}$ and $\mathbf{p}+\mathbf{u}$ in Euclidean 3-space goes exactly one circle, and its center is known (according to problem #8 in §9 of .../Cross.pdf) to lie at

$$\mathbf{c} := \mathbf{p} + \tfrac{1}{2}(\|\mathbf{v}\|^2\cdot\mathbf{u}\mathbf{u}^T - \|\mathbf{u}\|^2\cdot\mathbf{v}\mathbf{v}^T)\cdot(\mathbf{v}+\mathbf{u})/\big(\|\mathbf{v}\|^2\cdot\|\mathbf{u}\|^2 - (\mathbf{v}^T\mathbf{u})^2\big)\ .$$

From any four distinct concyclic points we may choose two subsets of three arbitrarily, and each subset must lie on the same circle and thus determine the same center via this formula. Let our two subsets have, say, …

$$\{\ \mathbf{p} := \mathbf{q} = \mathbf{o}\ ,\ \mathbf{u} := \mathbf{t}-\mathbf{q} = \mathbf{t}\ ,\ \mathbf{v} := \mathbf{q}-\mathbf{w} = -\mathbf{w}\ \}\quad \text{and}\quad \{\mathbf{p} := \mathbf{o}\ ,\ \mathbf{u} := \mathbf{t}\ ,\ \mathbf{v} := \mathbf{q}-\mathbf{z} = -\mathbf{z}\ \}\ .$$

Then substitution into the formula above provides an equation …

$$\big(\|\mathbf{z}\|^2\cdot\|\mathbf{t}\|^2 - (\mathbf{z}^T\mathbf{t})^2\big)\cdot(\|\mathbf{w}\|^2\cdot\mathbf{t}\mathbf{t}^T - \|\mathbf{t}\|^2\cdot\mathbf{w}\mathbf{w}^T)\cdot(\mathbf{t}-\mathbf{w}) = \big(\|\mathbf{w}\|^2\cdot\|\mathbf{t}\|^2 - (\mathbf{w}^T\mathbf{t})^2\big)\cdot(\|\mathbf{z}\|^2\cdot\mathbf{t}\mathbf{t}^T - \|\mathbf{t}\|^2\cdot\mathbf{z}\mathbf{z}^T)\cdot(\mathbf{t}-\mathbf{z})$$

that simplifies drastically, I hope.

Alternate approach:

Given three non-collinear points $\mathbf{p}-\mathbf{v}$, $\mathbf{p}$ and $\mathbf{p}+\mathbf{u}$ in Euclidean 3-space, their circle's center

$$\mathbf{c} := \mathbf{p} + \tfrac{1}{2}(\mathbf{v}^{¢}\cdot\mathbf{u})^{¢}\cdot(\mathbf{v}\cdot\|\mathbf{u}\|^2 + \mathbf{u}\cdot\|\mathbf{v}\|^2)/\|\mathbf{v}^{¢}\cdot\mathbf{u}\|^2\ .$$

Here "$\mathbf{v}^{¢}\cdot\mathbf{u}$" rewrites "$\mathbf{v}\times\mathbf{u}$" to turn cross-products into associative matrix multiplications with

Jacobi's Identity: $(\mathbf{v}^{¢}\cdot\mathbf{u})^{¢} = \mathbf{v}^{¢}\cdot\mathbf{u}^{¢} - \mathbf{u}^{¢}\cdot\mathbf{v}^{¢}$. Grassmann's Identity: $(\mathbf{v}^{¢}\cdot\mathbf{u})^{¢} = \mathbf{u}\cdot\mathbf{v}^T - \mathbf{v}\cdot\mathbf{u}^T$.

Substituting our subsets into the equation for $\mathbf{c}$ produces an equation …

$$(\mathbf{w}^{¢}\cdot\mathbf{t})^{¢}\cdot(\mathbf{w}\cdot\|\mathbf{t}\|^2 - \mathbf{t}\cdot\|\mathbf{w}\|^2)/\|\mathbf{w}^{¢}\cdot\mathbf{t}\|^2 = (\mathbf{z}^{¢}\cdot\mathbf{t})^{¢}\cdot(\mathbf{z}\cdot\|\mathbf{t}\|^2 - \mathbf{t}\cdot\|\mathbf{z}\|^2)/\|\mathbf{z}^{¢}\cdot\mathbf{t}\|^2$$

that Jacobi's Identity turns into

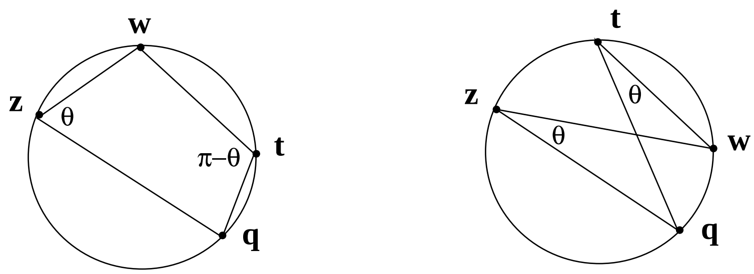
$$\mathbf{w}^{¢}\cdot\mathbf{t}^{¢}\cdot\mathbf{w}\cdot\|\mathbf{t}\|^2/\|\mathbf{w}^{¢}\cdot\mathbf{t}\|^2 - \mathbf{z}^{¢}\cdot\mathbf{t}^{¢}\cdot\mathbf{z}\cdot\|\mathbf{t}\|^2/\|\mathbf{z}^{¢}\cdot\mathbf{t}\|^2 = \mathbf{t}^{¢}\cdot\mathbf{z}^{¢}\cdot\mathbf{t}\cdot\|\mathbf{z}\|^2/\|\mathbf{z}^{¢}\cdot\mathbf{t}\|^2 - \mathbf{t}^{¢}\cdot\mathbf{w}^{¢}\cdot\mathbf{t}\cdot\|\mathbf{w}\|^2/\|\mathbf{w}^{¢}\cdot\mathbf{t}\|^2\ .$$

4: Given four distinct complex numbers $\mathbf{q}$, $\mathbf{t}$, $\mathbf{w}$ and $\mathbf{z}$ regarded as points in the plane, what condition do these numbers satisfy just when the points are concyclic? Why?

Solution: A condition both necessary and sufficient that the four given points lie on the same circle or straight line is that $\big((\mathbf{w}-\mathbf{z})\cdot(\mathbf{q}-\mathbf{t})\big)/\big((\mathbf{q}-\mathbf{z})\cdot(\mathbf{w}-\mathbf{t})\big)$ be a nonzero real number. Its sign depends upon the order of the points. Why and how will be explained next.

First the condition will be proved necessary: Suppose the given points lie on a circle or on a straight line regarded as closed by one point at ∞ . Two configurations are possible. The first has $\mathbf{q}$, $\mathbf{t}$, $\mathbf{w}$ and $\mathbf{z}$ in consecutive order around the circle; they are successive vertices of a concyclic quadrilateral whose opposite angles must be *Supplementary*, as shown first below.

Convex and Crossed Concyclic Quadrilaterals



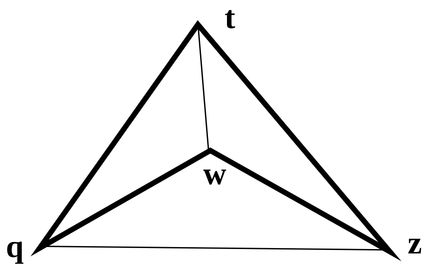
In this case, $(\mathbf{w}-\mathbf{z})/(\mathbf{q}-\mathbf{z}) = \alpha\cdot e^{\imath\theta}$ for some $\alpha > 0$ and $0 \le \theta \le \pi$, and $(\mathbf{q}-\mathbf{t})/(\mathbf{w}-\mathbf{t}) = \beta\cdot e^{\imath(\pi-\theta)}$ for some $\beta > 0$. Multiply to confirm that $\big((\mathbf{w}-\mathbf{z})\cdot(\mathbf{q}-\mathbf{t})\big)/\big((\mathbf{q}-\mathbf{z})\cdot(\mathbf{w}-\mathbf{t})\big) = \alpha\beta\cdot e^{-\imath\pi} = -\alpha\beta < 0$.

In the second configuration **q** and **t** separate **w** and **z**, as shown second above. (Swapping **z** and **w** changes the picture but not much of the proof.) Now $(\mathbf{w}-\mathbf{z})/(\mathbf{q}-\mathbf{z}) = \alpha\cdot e^{\imath\theta}$ and $(\mathbf{q}-\mathbf{t})/(\mathbf{w}-\mathbf{t}) = \beta\cdot e^{-\imath\theta}$ so that $\big((\mathbf{w}-\mathbf{z})\cdot(\mathbf{q}-\mathbf{t})\big)/\big((\mathbf{q}-\mathbf{z})\cdot(\mathbf{w}-\mathbf{t})\big) = \alpha\beta > 0$, confirming the condition.

To prove the condition sufficient, reverse the argument and, depending upon the sign of the real quotient, exploit either the fact that a convex quadrilateral whose opposite angles are supplementary must be concyclic, or else the fact that two triangles with a common base on the same side of their other vertices, where the angles are equal, must have the same circumcircle.

5: Given only the six *Euclidean distances* $||\mathbf{q}-\mathbf{t}||$, $||\mathbf{q}-\mathbf{w}||$, $||\mathbf{q}-\mathbf{z}||$, $||\mathbf{t}-\mathbf{w}||$, $||\mathbf{w}-\mathbf{z}||$ and $||\mathbf{z}-\mathbf{t}||$ between four distinct points in Euclidean n-space, what condition do these distances satisfy just when the points are concyclic? Why? (Looking up *Ptolemy's Inequality* may help.)

Solution: The condition is that the largest of the three products $||\mathbf{w}-\mathbf{z}||\cdot||\mathbf{t}-\mathbf{q}||$, $||\mathbf{z}-\mathbf{t}||\cdot||\mathbf{w}-\mathbf{q}||$ and $||\mathbf{w}-\mathbf{t}||\cdot||\mathbf{z}-\mathbf{q}||$ equal the sum of the other two. It comes from *Ptolemy's Inequality*, which says that $||\mathbf{w}-\mathbf{z}||\cdot||\mathbf{t}-\mathbf{q}|| + ||\mathbf{z}-\mathbf{t}||\cdot||\mathbf{w}-\mathbf{q}|| \ge ||\mathbf{w}-\mathbf{t}||\cdot||\mathbf{z}-\mathbf{q}||$ with equality just when the four points lie on the same circle or straight line and are so ordered that **q** and **z** separate **t** and **w**. A mnemonic for this inequality treats the four points as vertices of a tetrahedron on any closed path that traverses some four of the tetrahedron's six edges. The path is a quadrilateral; the sum of the products of its opposite edge-lengths exceeds the product of the two untraversed edge-lengths except when the tetrahedron has collapsed onto a concyclic quadrilateral or straight line segment (in a 2-dimensional plane in the n-dimensional space).



Ptolemy's Inequality:

$$||\mathbf{w}-\mathbf{z}||\cdot||\mathbf{t}-\mathbf{q}|| + ||\mathbf{z}-\mathbf{t}||\cdot||\mathbf{w}-\mathbf{q}|| \ge ||\mathbf{w}-\mathbf{t}||\cdot||\mathbf{z}-\mathbf{q}||$$

with equality just when the tetrahedron's vertices lie on the same circle or straight line so ordered that **q** and **z** separate **t** and **w**.

Problem 5's solution is a byproduct of a deduction of Ptolemy's inequality from the *Triangle Inequality* applied to *Inversion in the Unit Sphere*. Such an inversion is an invertible relation

$\mathbf{y} = \mathbf{x}/\|\mathbf{x}\|^2$ or equivalently $\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|^2$ wherein the Euclidean norm $\|\mathbf{u}\| := \sqrt{(\mathbf{u}'\cdot\mathbf{u})}$.

Either equation between $\mathbf{x}$ and $\mathbf{y}$ implies the other except perhaps when a vector is $\mathbf{o}$ or ∞ . Moreover the identity $\| \mathbf{x}/\|\mathbf{x}\|^2 - \mathbf{y}/\|\mathbf{y}\|^2 \| \equiv \| \mathbf{x} - \mathbf{y} \|/(\|\mathbf{x}\|\cdot\|\mathbf{y}\|)$ for arbitrary nonzero $\mathbf{x}$ and $\mathbf{y}$ is a quick consequence of the Euclidean norm's definition.

No generality is lost by a shift of origin $\mathbf{o}$ to $\mathbf{q}$, say, that subtracts $\mathbf{q}$ from each of the four given vectors. Doing so turns Ptolemy's inequality into the equivalent inequality

$$\|\mathbf{w}-\mathbf{z}\|\cdot\|\mathbf{t}\| + \|\mathbf{z}-\mathbf{t}\|\cdot\|\mathbf{w}\| \ \ge\ \|\mathbf{w}-\mathbf{t}\|\cdot\|\mathbf{z}\| \ .$$

Its confirmation begins with the Triangle inequality, which asserts that

$$\| \mathbf{w}/\|\mathbf{w}\|^2 - \mathbf{t}/\|\mathbf{t}\|^2 \| \ \le\ \| \mathbf{w}/\|\mathbf{w}\|^2 - \mathbf{z}/\|\mathbf{z}\|^2 \| \ +\ \| \mathbf{z}/\|\mathbf{z}\|^2 - \mathbf{t}/\|\mathbf{t}\|^2 \| \ .$$

Into this substitute the identity above and multiply by $\|\mathbf{t}\|\cdot\|\mathbf{w}\|\cdot\|\mathbf{z}\|$ to finish the confirmation.

The Triangle inequality above, and therefore Ptolemy's, becomes an equality just when $\mathbf{w}/\|\mathbf{w}\|^2 - \mathbf{z}/\|\mathbf{z}\|^2 = \mu\cdot(\mathbf{z}/\|\mathbf{z}\|^2 - \mathbf{t}/\|\mathbf{t}\|^2)$ for some $\mu > 0$; this is equivalent to the equation

$$\mathbf{z}/\|\mathbf{z}\|^2 \ =\ (\mathbf{w}/\|\mathbf{w}\|^2 + \mu\cdot\mathbf{t}/\|\mathbf{t}\|^2)/(1 + \mu) \ .$$

Think of μ as a variable parameter. As it increases from 0 to $+\infty$ it puts $\mathbf{z}/\|\mathbf{z}\|^2$ in a straight line segment running from $\mathbf{w}/\|\mathbf{w}\|^2$ to $\mathbf{t}/\|\mathbf{t}\|^2$. Inverting the inversion puts $\mathbf{z}$ into an arc that runs from $\mathbf{w}$ to $\mathbf{t}$. That arc is part of the image by inversion of a straight line. The image is a circle through $\mathbf{o}$ or, if the line passes through $\mathbf{o}$, the line itself with $\mathbf{o}$ and ∞ swapped. This assertion can be confirmed by restricting attention to a plane 2-dimensional subspace of the n-space containing $\mathbf{o}$ and the line: Choose orthonormal [x, y]-coordinates in which the line's equation is, say, $y = \eta$ (a constant) and observe that inversion's image $[x, y] = [\xi, \eta]/(\xi^2 + \eta^2)$ runs along a locus whose equation (deduced by eliminating the free parameter ξ) turns out to be $\eta\cdot(x^2 + y^2) - y = 0$, the equation of a circle through $[0, 0]$ or, if $\eta = 0$, the same straight line with $[0, 0]$ and $[\infty, 0]$ swapped.

After restoring the origin we infer that equality in Ptolemy's inequality is a condition necessary for four given distinct points $\mathbf{q}$, $\mathbf{t}$, $\mathbf{w}$ and $\mathbf{z}$ to lie on the same circle or straight line in such an order that $\mathbf{q}$ and $\mathbf{z}$ separate (or are separated by) $\mathbf{t}$ and $\mathbf{w}$. But problem 5 provides only the six distances between the points, not their order. Without order information, the condition necessary for our four points to be concyclic is that the largest of the three products

$$\|\mathbf{w}-\mathbf{z}\|\cdot\|\mathbf{t}-\mathbf{q}\| \ ,\ \ \|\mathbf{z}-\mathbf{t}\|\cdot\|\mathbf{w}-\mathbf{q}\| \ \text{ and } \ \|\mathbf{w}-\mathbf{t}\|\cdot\|\mathbf{z}-\mathbf{q}\|$$

equal the sum of the other two. This condition is also sufficient because the reasoning above is reversible.