Problem 1: Show that complex numbers $z = x + \imath y$ and special 2-by-2 matrices $Z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ are *isomorphic* (algebraically indistinguishable) at least so far as rational arithmetic operations $\{+, —, \cdot, \div\}$ are concerned. Pay close attention to division.

Solution 1: Obtain $\bar{Z}$ from Z in the same way as conjugate $\bar{z}$ is obtained from z, namely by reversing the sign of y. Then $Z^{-1} = \bar{Z}/|z|^2$ just as $1/z = \bar{z}/|z|^2$. Consequently every rational operation $t := w¤z$ performed upon complex variables maps turns out to match the analogous operation $T := W¤Z$ upon special matrices provided " W/Z " is construed as " $W\cdot Z^{-1}$ ".

Problem 2: It is tautologically true that "Complex variables are unnecessary because anything that can be done with them can also be done without them." However this is not the whole truth; it omits mention of the extra cost of complications incurred if our thoughts must be formulated in terms solely of real variables with no mention of complex. We have already seen that one complex division $w/z = (u + \imath v)/(x + \imath y) = (u\cdot x + v\cdot y)/(x^2 + y^2) + \imath(v\cdot x - u\cdot y)/(x^2 + y^2)$ entails six real multiplications, three real additions and two real divisions; these are far more complicated than the concept of complex division.

Now consider the complex Principal Square Root $u + \imath v = w = \sqrt{z}$, which maps the whole z-plane 1-to-1 onto the right-hand half of the w-plane including its positive imaginary axis (if not all of this axis). $\sqrt{z}$ must be discontinuous as z crosses a *Slit* along the negative real axis, because the inverse map $x + \imath y = z = w^2$ takes tiny semicircular neighborhoods around both boundary points $\pm\imath v$ of the right half of the w-plane to a roughly circular neighborhood around $x = -v^2 < 0$ in the z-plane. Given real values for x and y, how should u and v be computed using just two (not three) real square roots and a few rational (not transcendental) real arithmetic operations and (because of the discontinuity) comparisons? Hint: $|w| = \sqrt{|z|}$.

Solution 2: The computed values u and v must satisfy $u \ge 0$, $x = u^2 - v^2$ and $y = 2uv$. Consequently sign(y) = sign(v) when they are nonzero, even if $u = 0$. Here is the algorithm:

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u := √(( |x| + √(x² + y²) )/2) ;
If u = 0 then { v := y }
    else if x < 0 then { v := CopySign(y, u) ; u := y/(2v) }
        else { v := y/(2u) } .
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(CopySign(y, u) here has the magnitude of u but the same sign bit as y has. Alas, other definitions of CopySign reverse its arguments to remain "Compatible" with the SIGN(u, y) function provided by ancient Fortran programming languages.)

Note that this algorithm, unlike others less carefully designed, copes properly with the special cases $z = 0$ and $z = x < 0$. The algorithm's use of $|x|$ to first compute u avoids cancellation in case $y \approx 0 > x$. Then the test " $u = 0$ " precludes subsequent division by zero. The test " $x < 0$ " compensates for the earlier use of $|x|$, and thus avoids the need to compute a third square root $\sqrt{((-|x| + \sqrt{x^2 + y^2})/2)}$, which would suffer from roundoff anyway. Then the test implicit in the CopySign function implements the complex square root's discontinuity accurately despite roundoff when y is extremely tiny. Except across that discontinuity, where $x < 0$ and y reverses sign, this algorithm for $\sqrt{x + \imath y}$ is obviously continuous in x and y. (A computer implementation must be complicated by measures taken to avoid premature exponent over/underflow when u is first computed.)

Problem 3: For any nonzero complex constant $ç$ define $f(z) := ç\cdot\sqrt{(z/ç^2)}$. Apparently $f(z)^2 = z$ for all arguments z but $f(z) = +\sqrt{z}$ for some and $f(z) = -\sqrt{z}$ for others; which?

Solution 3: $f(0) = +\sqrt{0} = +0$. Henceforth assume $z \ne 0$; let $Ø := \arg(z)$ and $Ç := \arg(ç)$, so $-\pi < Ø \le \pi$ and $-\pi < Ç \le \pi$. Now $f(z)/\sqrt{z} = \pm 1$ depending upon $Ø$ and $Ç$, not upon $|z|$ nor $|ç|$, thus:

$$\arg(f(z)/\sqrt{z}) = \big((Ø-2Ç \bmod 2\pi)/2 - (Ø-2Ç)/2\big) \bmod 2\pi \quad \text{where in general}$$

$$\begin{aligned} H \bmod 2\pi \ &:= H && \text{if } -\pi < H \le \pi , \\ &:= H - 2\pi && \text{if } \ \pi < H \le 3\pi , \text{ or} \\ &:= H + 2\pi && \text{if } -3\pi < H \le -\pi . \end{aligned}$$

Consequently
$$\begin{aligned} \big((H \bmod 2\pi)/2 - H/2\big) \bmod 2\pi \ &= 0 && \text{if } -\pi < H \le \pi , \\ &= \pi && \text{if } -3\pi < H \le \pi \text{ or } \pi < H \le 3\pi . \end{aligned}$$

For our purposes $-3\pi < H := Ø-2Ç \le 3\pi$, whence comes a table showing the ranges of $Ø$ over which $f(z) = \pm\sqrt{z}$:

$Ø = \arg(z)$	$-\pi < Ç = \arg(ç) \le 0$	$0 \le Ç = \arg(ç) \le \pi$
$f(z) = -\sqrt{z}$	$2Ç+\pi < Ø \le \pi$	$-\pi < Ø \le 2Ç-\pi$
$f(z) = +\sqrt{z}$	$-\pi < Ø \le 2Ç+\pi$	$2Ç-\pi < Ø \le \pi$

I.e., $f(z)$ is a square root of z discontinuous across a slit along $\arg(z) = 2Ç - \mathrm{CopySign}(Ç, \pi)$.

Problem 4: Although $\sqrt{p\cdot q} = \sqrt{p}\cdot\sqrt{q}$ for all nonnegative real p and q , the principal square root of complex arguments p and q does not always satisfy that identity.

a) For which arguments p and q does $\sqrt{p\cdot q} \ne \sqrt{p}\cdot\sqrt{q}$? What happens instead?

b) For which complex arguments z does $\sqrt{(1 - z^2)} = \sqrt{1-z}\cdot\sqrt{1+z}$?

c) For which complex arguments w does $\sqrt{(w^2 - 1)} = \sqrt{w-1}\cdot\sqrt{w+1}$?

(This problem is very hard if approached naively.)

Solution 4(a): $\sqrt{p\cdot q} = -\sqrt{p}\cdot\sqrt{q}$ unless $-\pi < \arg(p) + \arg(q) \le \pi$. Why? See Solution 3.

Solution 4(b): $\sqrt{(1 - z^2)} = \sqrt{1-z}\cdot\sqrt{1+z}$ for all z . Why? Compare derivatives.

Solution 4(c): $\sqrt{(w^2 - 1)} = \sqrt{w-1}\cdot\sqrt{w+1}$ just when $\mathrm{Re}(w) > 0$ or $\mathrm{Re}(w) = 0 \le \mathrm{Im}(w)$.

Why? The neat proof appeals to the *Monodromy* Theorem, which will be taken up later.

Problem 5: £ is the straight line through two given points b and $b+ç$ in the complex z-plane.

(a) What equation must z satisfy when it lies on £ ?

(b) Find a formula for $|z - £|$, the shortest distance from z to £ if z is not on £ .

(c) Given a point f not on £ , " $|z-f| = |z-£|$ " is the equation of a *Parabola* whose *Focus* is f and *Directrix* is £ . Obtain an equivalent equation in terms exclusively of $z, b, ç$ and f without the square root implicit in $|\ldots|$.

Solution 5(a): $\mathrm{Im}((z-b)/ç) = 0$, or equivalently $\mathrm{Im}(\bar{ç}\cdot(z-b)) = 0$.

Solution 5(b): $|z-£| = |ç\cdot\mathrm{Im}((z-b)/ç)|$ because the point on £ nearest z is $z - \imath ç\cdot\mathrm{Im}((z-b)/ç)$.

Solution 5(c): $|(z-f)/ç|^2 = \mathrm{Im}((z-b)/ç)^2$, or equivalently $|\bar{ç}\cdot(z-f)|^2 = \mathrm{Im}(\bar{ç}\cdot(z-b))^2$.

Solutions for Problems 6 and 7 can be found in the class notes on Conic Sections in the Complex z-plane.