Angle subtended at the eye by neighboring stars W. Kahan

Spherical polar coordinates $\{R, H, A\}$ convert to Cartesian (X, Y, Z) :

R = radius, H = angle of Elevation above horizon, A = angle of Azimuth from compass North.

$$R > 0\ , \quad -\pi/2 \le H \le \pi/2\ , \quad -\pi \le A \le \pi\ .$$

X = distance North, Y = distance West, Z = distance Up ;

$$X = R \cdot \cos H \cdot \cos A\ , \qquad Y = R \cdot \cos H \cdot \sin A\ , \qquad Z = R \cdot \sin H\ .$$

Neighboring stars: Put one at $(X, Y, Z) \longleftrightarrow \{ R\ ,\ H\ ,\ A \}$ for any arbitrary $R > 0$;

$$X = R\cdot\cos H \cdot \cos A\ , \quad Y = R\cdot\cos H \cdot \sin A\ , \quad Z = R\cdot\sin H\ ;$$

another at $(X+x, Y+y, Z+z) \longleftrightarrow \{ R\ ,\ H+h\ ,\ A+a \}$;

$$X+x = R\cdot\cos(H+h)\cdot\cos(A+a)\ , \quad Y+y = R\cdot\cos(H+h)\cdot\sin(A+a)\ , \quad Z+z = R\cdot\sin(H+h)\ .$$

The angle v subtended at the eye by the stars satisfies $(2\cdot R\cdot\sin v/2)^2 = x^2 + y^2 + z^2$, so

$$\begin{aligned}(2\cdot\sin v/2)^2 &= (\cos(H+h)\cdot\cos(A+a) - \cos H \cdot \cos A)^2 + (\cos(H+h)\cdot\sin(A+a) - \cos H \cdot \sin A)^2 + (\sin(H+h) - \sin H)^2 \\ &= 2 - 2\cdot\sin(H+h)\cdot\sin H - 2\cdot\cos(H+h)\cdot\cos(H)\cdot\cos(a) \\ &= 2\cdot(1 - \cos(h)) + 2\cdot(1 - \cos a)\cdot\cos(H+h)\cdot\cos(H) \\ &= 4\cdot\sin^2(h/2) + 4\cdot\sin^2(a/2)\cdot\cos(H+h)\cdot\cos(H)\ .\end{aligned}$$

Conclusion: Of the two azimuths A and $A+a$ only their difference $a \bmod 2\pi$ matters; then

$$\begin{aligned} v &= \arccos(\sin(H+h)\cdot\sin(H) + \cos(H+h)\cdot\cos(H)\cdot\cos(a)) \\ &= 2\cdot\arcsin\sqrt{\left(\sin^2(h/2) + \sin^2(a/2)\cdot\cos(H+h)\cdot\cos(H)\right)}\ .\end{aligned}$$

The first formula malfunctions at small subtended angles. The second is numerically fine at small angles v but loses almost half the precision carried if $v \approx \pi$ though no cancellation occurs.

Example: $a = 179.999^\circ$, $h = 52^\circ$, $H = -26^\circ$; carrying 10 sig. dec., this $v = 180^\circ$ instead of 179.999101° .

We can do better. Since $-\pi/2 \le H \le \pi/2$ and $-\pi/2 \le H+h \le \pi/2$,

$$\begin{aligned} 0 \le \cos(H+h)\cdot\cos(H) = (\cos(2H+h) + \cos(h))/2 &= \cos^2(H+h/2) - \sin^2(h/2) \\ &= \cos^2(h/2) - \sin^2(H+h/2)\ .\end{aligned}$$

Therefore

$$\begin{aligned}\tan^2(v/2) &= \sin^2(v/2)/(1 - \sin^2(v/2)) \\ &= \left(\sin^2(h/2) + \sin^2(a/2)\cdot\cos(H+h)\cdot\cos(H)\right)/\left(\cos^2(h/2) - \sin^2(a/2)\cdot\cos(H+h)\cdot\cos(H)\right) \\ &= \left(\sin^2(h/2)\cdot\cos^2(a/2) + \sin^2(a/2)\cdot\cos^2(H+h/2) \right)/\left(\cos^2(h/2)\cdot\cos^2(a/2) + \sin^2(a/2)\cdot\sin^2(H+h/2) \right)\ .\end{aligned}$$

Hence follows a numerically accurate and efficient formula:

$$v = 2\cdot\arctan\sqrt{\left(\left(th\cdot(1 + ta + TH) + ta \right)/\left(1 + TH\cdot(1 + ta\cdot(1+th))\right)\right)}$$

wherein $ta = \tan^2(a/2)$, $th = \tan^2(h/2)$ and $TH = \tan^2(H+h/2)$. Only if $h = 0$ and $H = \pm\pi/2$ does $TH = \infty$. Therefore presubstitute $0\cdot\infty = 0$ and $\infty/\infty = 1/TH$ to handle *all* special cases. (Actually, infinite floating-point values of $\tan(\dots)$ can't arise unless angles are in degrees.)

The same formula works with astronomical Declination instead of Elevation and Right Ascension instead of Azimuth. A similar formula would work for distance over the surface of the Earth, using Latitude instead of Elevation and Longitude instead of Azimuth, if the Earth were perfectly spherical.