The **Taylor Series** of an infinitely differentiable vector-valued function $y(t)$ of a scalar $t$ is

$$y(t) = y(0) + t \cdot y'(0) + \frac{t^2}{2} y''(0)/2 + t^3 y'''(0)/6 + t^4 y^{(4)}(0)/24 + \ldots$$

These derivatives can be computed for a solution of the **Initial Value Problem**

$$y(0) = y_0 \text{ is given, and } y'(t) = f(y(t)) \text{ for all } t \geq 0$$

from the derivatives of the given vector-valued function $f(y)$. In fact, from the **Chain Rule**,

$$y' = f' \cdot f, \quad y'' = f'' \cdot f + f^2 \cdot f', \quad y''' = f''' \cdot f \cdot f + 3 f'' \cdot f' \cdot f + f' \cdot f'' \cdot f + f^2 \cdot f'' + f^3 \cdot f, \ldots$$

Then each derivative of $y(t)$ can be evaluated at $t = 0$ by evaluating each derivative of $f(y)$ at $y = y(0) = y_0$. Note that the higher derivatives of $f$ are symmetric multilinear operators; for instance, $f''(y)$ is a symmetric bilinear operator: $f''(u, v) = f''(v, u)$ is a vector-valued linear function of each vector $u$ and $v$ separately. Because linear operators do not necessarily commute, $f'''(y)$ is a symmetric multilinear operator: $f'''(y)$ is a vector-valued linear function of each vector $y$. Hence, the computed approximation

$$y(t+h) = y(t) + h \cdot y'(t) + h^2 \cdot y''(t)/2 + h^3 \cdot y'''(t)/6 + h^4 \cdot y^{(4)}(t)/24 + \ldots$$

in which the derivatives $dy(t+h)/dt$ etc. are computed at $t = 0$ from the same formulas

$$y' = f, \quad y'' = f' \cdot f, \quad y''' = f'' \cdot f + f^2 \cdot f', \quad y''' = f''' \cdot f \cdot f + 3 f'' \cdot f' \cdot f + f' \cdot f'' \cdot f + f^2 \cdot f'' + f^3 \cdot f, \ldots$$

as before except that now $f(y)$ and its derivatives are computed at $y = y(t)$.

Normally the Taylor series would be used to obtain $y(t+h)$ from $y(t)$ for any sufficiently small stepsize $h$:

$$y(t+h) = y(t) + h \cdot y'(t) + h^2 \cdot y''(t)/2 + h^3 \cdot y'''(t)/6 + h^4 \cdot y^{(4)}(t)/24 + \ldots$$

A similar process generates a formal series for any one-step numerical method’s formula that advances an approximate solution $y = y(t)$ through one step $h$ to $Y = Y(t+h)$, but now we differentiate with respect to $h$ instead of $t$ to get

$$Y(t+h) = y(t) + h \cdot y'(t) + h^2 \cdot Y''(t)/2 + h^3 \cdot Y'''(t)/6 + h^4 \cdot Y^{(4)}(t)/24 + \ldots$$

in which $y = y(t)$ and the derivatives $Y'$ etc. are derivatives of $Y(t+h)$ with respect to $h$ evaluated at $h = 0$. These derivatives depend upon the numerical method’s formula. For example, take the (implicit) **Trapezoidal Rule** $Y = y + h \cdot (f(y) + f(Y))/2$. Now, at $t+h$,

$$Y' = (f(y) + f(Y))/2 + h \cdot f'(Y) \cdot Y'/2,$$

$$Y'' = f'(Y) \cdot Y' + h \cdot (f''(Y) \cdot Y' + f'(Y) \cdot Y'')/2,$$

$$Y''' = 3 f''(Y) \cdot Y' \cdot Y'/2 + f'(Y) \cdot Y'''/2 + h \cdot (f'''(Y) \cdot Y' \cdot Y' + \ldots)/2,$$

etc. Here every instance of $Y$ or its derivatives is evaluated at $t+h$. For the Taylor series we set $h = 0$ in the foregoing formulas and substitute for derivatives of $Y$ in succession to get

$$y' = f, \quad y'' = f' \cdot f, \quad y''' = (3f'' \cdot f + f^2) \cdot f/2, \quad \ldots$$

in which now the derivatives of $Y$ are evaluated at $t$, and $f$ and its derivatives are evaluated at $y(t)$. Hence, the computed approximation

$$y(t+h) = y(t) + h \cdot f + h^2 \cdot f \cdot f/2 + h^3 \cdot (3f'' \cdot f + f^2) \cdot f/12 + \ldots$$

can be compared with the local solution

$$y(t+h) = y(t) + h \cdot f + h^2 \cdot f \cdot f/2 + h^3 \cdot (f'' \cdot f + f^2) \cdot f/6 + \ldots$$

to reveal the 2nd-order Trapezoidal Rule’s local truncation error

$$y(t+h) - Y(t+h) = h^3 (f^2 - f' \cdot f) \cdot f/12 + \ldots.$$
Another example is the (implicit) **Midpoint Rule** \( Y = y + h \cdot f((y+Y)/2) \). At \( t+h \),
\[
\begin{align*}
Y' &= f((y+Y)/2) + h \cdot f'((y+Y)/2) \cdot Y'/2 , \\
Y'' &= f'((y+Y)/2) \cdot Y' + h \cdot ( f''((y+Y)/2) \cdot Y'/4 + f'((y+Y)/2) \cdot Y''/2 ) , \\
Y''' &= 3f''((y+Y)/2) \cdot Y' \cdot Y'/4 + 3f'((y+Y)/2) \cdot Y''/2 + h \cdot ( f''' \ldots ) , \ 	ext{etc.}
\end{align*}
\]
Setting \( h = 0 \) and substituting for derivatives of \( Y \) in succession yields
\[
\begin{align*}
Y' &= f , \ 
Y'' &= f' \cdot f , \ 
Y''' &= ( 3f'' \cdot f + 6f'^2 ) \cdot f/4 , \ 
\ldots
\end{align*}
\]
in which now the derivatives of \( Y \) are evaluated at \( t \), and \( f \) and its derivatives are evaluated at \( y(t) \). Hence, by comparison with the Taylor series for \( y(t+h) \) we find the 2nd-order Midpoint Rule’s local truncation error to be
\[
y(t+h) - Y(t+h) = h^3 \cdot ( f'' \cdot f - 2f'^2 ) \cdot f/24 + \ldots
\]

The foregoing manipulations are tedious enough that only a computerized algebra system should perform them. However, programming **Maple** or **Mathematica** or **Derive** to perform them has been more difficult than it should be. At least some of the difficulty arises, I think, because these languages disallow declaration of a variable’s linguistic **Type**. Besides the derivatives’ non-commutative partially associative multiplication, their multi-linear symmetry has to be taken into account in order to achieve correct simplifications of expressions like the fifth derivative
\[
y^5 = f''' \cdot f \cdot f \cdot f \cdot f + 6f''' \cdot f' \cdot f \cdot f \cdot f + 4f''' \cdot f \cdot f \cdot f + 4f''' \cdot f \cdot f +
+ 3f'' \cdot f' \cdot f' \cdot f + f'' \cdot f' \cdot f' \cdot f + 3f' \cdot f'' \cdot f' \cdot f + 3f'' \cdot f' \cdot f + f'' \cdot f' \cdot f + f'' \cdot f.
\]
( I hope I’ve gotten it right.)

The foregoing **formal** (because their convergence is undetermined) Taylor series expansions do not reveal an important property possessed by the computed solution \( Y \) of the Initial Value Problem when it is obtained from a Reflexive formula, which is a formula in which \( Y(t+h) \) and \( y(t) \) are merely swapped when the sign of \( h \) is reversed. The Midpoint and Trapezoidal Rules’ formulas are reflexive. The composition of \( T/h \) steps of a reflexive formula to approximate the true solution \( y(T) \) at a fixed \( T \), but using any sufficiently small stepsize \( h \) so long as \( T/h \) is an integer, can be proved to produce a computed approximation \( Y(T) \) that depends upon \( h \) and differs from \( y(T) \) by an error
\[
y(T) - Y(T) = c_2 h^2 + c_4 h^4 + c_6 h^6 + \ldots
\]
whose formal expansion in powers of \( h \) contains only even powers. The expansion need not converge for any \( h > 0 \); instead it is an Asymptotic expansion whose behavior is conveyed by its first few terms ever more accurately as \( h \to 0 \). Recomputations of \( Y(T) \) with diminishing stepsizes \( h, h/2, h/4, h/8, \ldots \) provides a sequence to which Richardson’s Extrapolation can be applied, as in Romberg integration, to achieve what amounts to higher-order convergence.