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A Survey on Model Reduction Methods to Reduce Degrees of Freedom of Linear Damped Vibrating Systems

ABSTRACT

This report describes the details of the model reduction methods to reduce degrees of freedom for the dynamic analysis of general linear and damped vibrating systems. The report starts with an introduction of an equation of motion of a two-story building—an example of a damped vibrating system. The responses of the similar equations of motion of damped systems are studied through eigenvalue problems. The force-dependent mode shape, quasi-static mode shape, and Rayleigh Ritz methods are applied to obtain some approximate mode shapes corresponding to the lowest undamped frequencies of the large systems. This is because the lowest undamped frequency modes are a lot more important to the high and undamped frequency modes in structure engineering. These the lowest undamped frequency mode shapes can then be used to transform the large systems into the smaller ones containing the transformed coordinates corresponding to the lowest undamped frequency modes of the large systems. These model reduction methods determine the approximations from only some low frequency modes thus it helps reduce time and cost of computation of the responses.

1. INTRODUCTION

The analysis of structures for dynamic excitation is dictated by the complexity of the structures, and several hundred to a few thousand degrees of freedom may be necessary for the accurate evaluation of the forces in the complex structures. The refined modeling can be used for dynamic analysis of the system, but it may be unnecessary and fewer degrees of freedom could be enough. That is the case because the dynamic response of many systems can be represented by the first few natural vibration modes, these modes can be determined accurately with significantly fewer degrees of freedom than required for static analysis. Thus we are interested in reducing the number of degrees of freedom as much as possible before proceeding with computation of natural frequencies and modes.

First, we need to start with the linear damped equations of motion of the structures to be studied. The equations of motion will be second order differential equations in matrix forms with the mass, damping and stiffness matrices as their coefficients. Without the damping term in the equations, the response of these equations can be produced by solving the eigenvalue problems and using the modal matrix to decouple the systems into modal equations. This approach is very efficient for undamped systems and systems with special types of damping e.g. proportional damping where the damping matrix is a linear combination of the mass matrix and the stiffness matrix. In the case of a damped system, the linear transformation is applied to the original equation of

motion to transform the mass matrix into a diagonal matrix. The approximate response can be determined by means of solving the decoupled equations of motion by neglecting the off-diagonal elements of the modal damping matrix and the modal stiffness matrix. This concept still works if the reduced equations of motion are needed. The model reduction method is applied to the original equation to get the some of the approximate lowest frequency mode shapes which will be formed as a transition matrix. Then the original equation is then transformed into a reduced equation at the same time as the mass matrix is changed into a diagonal matrix. The reduced system obtained can then be solved for the responses with less work than solving the original system.

The model reduction methods, whereby the number of degrees of freedom in the system is reduced, is applied to large system to give faster computation of the natural frequencies and mode shapes of a structure. There are many different iterative procedures for computing the eigenvectors or so called mode shapes corresponding to a chosen set of *n* eigenvalues of a symmetric matrix, e.g. subspace iteration, the Lanczos method and the trace minimization method. A classical method for computing approximate eigenvectors is the subspace iteration, a method to handle more than one vector.

Force-dependent mode shape vector method [2] has long been used to approximate the dynamic response of structures and as a model reduction technique to reduce the size of large-scale systems. Provided that the approximate mode shapes span the same configuration space, they are an attractive alternative to the conventional eigenvector method (normal mode method), since the mode shape vectors of the reduced systems can be computed with significantly less computational effort. Force-dependent mode shape vectors are a particular group vectors in which the information about the loading on the structure is used to generate vectors. The conventional mode shape vector method employs static recurrence procedures to generate the approximate modes shapes, which satisfies the static completeness condition. The force-dependent mode shape vector method has similar advantages and disadvantages as the mode acceleration method. Consequently, this method is best suited for relatively low-frequency problems. For higher-frequency, or banded frequency problems, large sets of mode shape vectors are needed to span the configuration space associated with the high frequencies. This will decrease the force-dependent mode shape vector method's efficiency. It also results in the loss of orthogonality of the mode shape vectors, which causes numerical errors in solving the reduced system.

An appropriate set of mode shape vectors should satisfy two conditions. First, the basis formed by mode shape vectors should be complete with respect to forcing (loading) patterns of the problem, at least for a frequency of interest. The force-dependent mode shape vectors satisfy this condition. Second, the mode shape vectors should span all desired frequency space. Normal modes always meet this condition since they consist of all modes in the frequency range. However, the force-dependent mode shape vectors do not satisfy this condition.

A new force-dependent mode shape vectors method called quasi-static mode shape vector method is introduced in [2] to satisfy both conditions above. This method

employs a quasi-static recurrence procedure based upon a new modal superposition technique. Comparing with the conventional force-dependent mode shape vectors method, this method is more efficient and more accurate (in terms of errors).

The subspace iteration method is then introduced to get some mode shapes simultaneously. To further improve the subspace iteration approximation, one can use the Rayleigh-Ritz procedure [7]. This is a well known method to increase the accuracy in the estimates, by little extra work. The Ritz values and Ritz vectors obtained with this procedure are the optimal approximations knowing some of the approximate mode shapes alone.

Deriving Equations of Motion of a Two-story Building Model

The structure dynamics problem can be formulated for structures discretized as systems with a finite number of degrees of freedom. The equations of motion are considered as a multi-degree-of-freedom system; e.g. a two storey frame that is subjected to external forces or earthquake. The equations of motion are applied to the buildings and the response will be analyzed.

A two-story building is a very simple multi degree of freedom system. The building frame is subjected to external forces $p_1(t)$ and $p_2(t)$ in the figure. The beams and floors are assumed to be rigid. The axial deformation of the beams and columns are neglected. The mass is idealized as it concentrates at the floor level because most of the building mass is at the floor level. Strictly speaking the mass is actually distributed throughout the building. The number of degrees of freedom is the number of independent coordinates that is required to define the positions of all the masses. The model of a two-story building is shown in Figure with mass at each floor, the lateral displacement x_1 and x_2 of the floors.

With Newton's Law of Motion, it gives the following equation for each mass:

$$m_j \ddot{x}_j(t) + f_{Dj}(t) + f_{Sj}(t) = p_j(t) \qquad (1)$$

where m_j are the masses on the j^{th} floor; $p_j(t)$, $f_{Dj}(t)$, $f_{Sj}(t)$ are the external forces, the elastic resisting forces and the damping forces respectively.

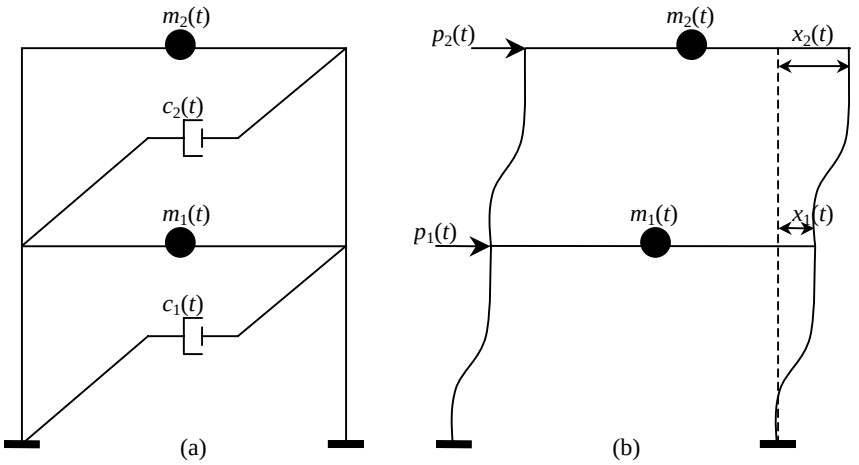

Figure 1: (a) Two-story frame without forces and (b) Two-story frame with forces

There are two equations of motion for the Figure 1 when j = 1 and 2, they can be written in the matrix form as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} + \begin{Bmatrix} f_{D1}(t) \\ f_{D2}(t) \end{Bmatrix} + \begin{Bmatrix} f_{S1}(t) \\ f_{S2}(t) \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{2}$$

or

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{f}_D(t) + \mathbf{f}_S(t) = \mathbf{p}(t) \tag{3}$$

where

$$\ddot{\mathbf{x}} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{f}_D(t) = \begin{Bmatrix} f_{D1}(t) \\ f_{D2}(t) \end{Bmatrix},$$
$$\mathbf{f}_S(t) = \begin{Bmatrix} f_{S1}(t) \\ f_{S2}(t) \end{Bmatrix}, \quad \mathbf{p}(t) = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Assuming the elastic resisting force f_S is linear; it is related to the floor displacement.

$$f_S = kx \tag{4}$$

where k is the lateral stiffness depending on the story height and a column with modulus and second moment of inertia.

With the stiffness defined and the Newton's laws of motion applied, the elastic resisting forces f_{S1} and f_{S2} are related to the floor displacements as follows:

$$f_{S1} = k_1 x_1 + k_2 (x_1 - x_2)$$
$$f_{S2} = k_2 (x_2 - x_1) \tag{5}$$

In the matrix form:

$$\mathbf{f}_S(t) = \begin{Bmatrix} f_{S1}(t) \\ f_{S2}(t) \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \tag{6}$$

or

$$\mathbf{f}_S(t) = \mathbf{K}\mathbf{x}(t) \tag{7}$$

The damping forces f_{D1} and f_{D2} are related to the floor velocities $\dot{x}_1$ and $\dot{x}_2$.

$$f_D = c\dot{x} \tag{8}$$

where c is the damping coefficient.

In the similar manner as Equation (5), we have

$$\mathbf{f}_D(t) = \begin{Bmatrix} f_{D1}(t) \\ f_{D2}(t) \end{Bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \tag{9}$$

or

$$\mathbf{f}_D(t) = \mathbf{C}\dot{\mathbf{x}}(t) \tag{10}$$

The Equations (7) and (10) are substituted into Equation (3) to obtain the following equation that is the general equation of motion of a linear vibratory system

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t) \tag{11}$$

where the initial conditions are $\mathbf{x}(t=0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(t=0) = \dot{\mathbf{x}}_0$; **M** is the mass matrix, **C** is the damping matrix, and **K** is the stiffness matrix. All of them are of order $N \times N$. The displacement $\mathbf{x}(t)$ and the external excitation $\mathbf{p}(t)$ are N-dimensional vectors. In the case of passive systems, which only have passive elements, **M**, **C**, and **K** matrices are all real, symmetric and positive definite.

2. THE EIGENVALUE PROBLEM

The equations of motion in the form of Equation (11) are linear second order differential equations. In order to solve these systems, it is easier to solve the similar

equations but without the damping term for the natural frequencies and natural modes first. And then apply those results to the original systems to get the approximate responses. In the absence of viscous damping and external force, the system in Equation (11) would be

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = 0 \tag{12}$$

An important case in the study of vibrations is that in which all coordinates have the same motion in time. To examine the possibility that such motions exist, we consider a solution of Equation (12) in the exponential form

$$\mathbf{x}(t) = e^{st}\phi \tag{13}$$

Introducing Equation (13) into Equation (12) and dividing through by e^{st}, we can write

$$\mathbf{K}\phi = \lambda\mathbf{M}\phi \qquad \text{where } \lambda = -s^2 \tag{14}$$

In order to find the vibration properties; such as, the undamped natural frequencies and modes of the system, the solution of the matrix eigenvalue problem (14) is required. Let ϕ_r (r = 1…N) be the eigenvectors of the generalized symmetric eigenvalue problem:

$$\mathbf{K}\phi_r = \lambda_r\mathbf{M}\phi_r \qquad \text{for} \quad r = 1...N \tag{15}$$

or

$$\mathbf{K}\phi_r = \omega_r^2\mathbf{M}\phi_r \tag{16}$$

where $\lambda_r \equiv \omega_r^2$ are the eigenvalues associated with the eigenvectors ϕ_r. Physically the vector ϕ_r is the r^{th} natural mode while ω_r is the r^{th} natural frequency of vibration. Let $\mathbf{\Phi}$ denote the $N \times N$ modal matrix associated with system (12). The columns of $\mathbf{\Phi}$ are the eigenvectors ϕ_r of the original system in Equation (12).

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_N \end{bmatrix} \tag{17}$$

The eigenvalues $\lambda_r \equiv \omega_r^2$ are the roots of the characteristic equation

$$p(\lambda_r) = \det[\mathbf{K} - \lambda_r\mathbf{M}] = 0 \tag{18}$$

where $p(\lambda_r)$ is a polynomial of order N, the number of degrees of freedom of the system. Note that this method is not practical for the large systems (large number of degrees of freedom) because it requires much work to evaluate the N coefficients and the numerical round-off errors might be significant.

The orthogonality of the modes

The natural modes corresponding to different natural frequencies can be shown to satisfy the following orthogonality conditions. When $\lambda_n \neq \lambda_r$

$$\phi_n^T \mathbf{K} \phi_r = 0 \tag{19}$$

$$\phi_n^T \mathbf{M} \phi_r = 0 \tag{20}$$

Furthermore, it can be shown that $\mathbf{\Phi}$ is real and nonsingular. The modal matrix $\mathbf{\Phi}$ is usually normalized according to

$$\mathbf{\Phi}^{\mathrm{T}} \mathbf{M} \mathbf{\Phi} = \mathbf{I} \tag{21}$$

where $\mathbf{\Phi}^{\mathrm{T}}$ is the transpose of $\mathbf{\Phi}$ and $\mathbf{I}$ is the identity matrix. Moreover,

$$\mathbf{\Phi}^{\mathrm{T}} \mathbf{K} \mathbf{\Phi} = \mathbf{\Lambda} = \mathrm{diag}[\lambda_1, ..., \lambda_N] = \mathrm{diag}[\omega_1^2, ..., \omega_N^2] \tag{22}$$

is the $N \times N$ matrix named a spectral matrix whose diagonal entries are the squares of the system's natural frequencies; i.e. ω_i^2. From Equations (15) and (16), we have

$$\mathbf{K} \mathbf{\Phi} = \mathbf{M} \mathbf{\Phi} \mathbf{\Lambda} \tag{23}$$

By applying a linear modal transformation to Equation (11) with

$$\mathbf{x}(t) = \mathbf{\Phi} \mathbf{q}(t), \tag{24}$$

Equation (11) is normalized to

$$\mathbf{I}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{\Lambda} \mathbf{q}(t) = \mathbf{f}(t) \tag{25}$$

where $\mathbf{f}(t) = \mathbf{\Phi}^{\mathrm{T}} \mathbf{p}(t)$ and $\mathbf{q}(t)$ is the vector of normal coordinates (or N-dimensional modal displacement vector). $\mathbf{D} = \mathbf{\Phi}^T \mathbf{C} \mathbf{\Phi}$ is called the modal damping matrix and is symmetric.

When $\mathbf{D}$ is diagonal, Equation (25) is a set of N decoupled, second-order differential equations, which can be solved independently of the others. Thus, we have

$$\ddot{q}_i(t) + d_i \dot{q}_i(t) + \omega_i^2 q_i(t) = f_i(t) \qquad i = 1 \ldots N \tag{26}$$

where d_i is the i^{th} diagonal element of matrix $\mathbf{D}$, $f_i(t)$ is the i^{th} component of the modified forcing vector.

However, the modal damping matrix **D** is usually not diagonal. Equation (26) is then coupled by the non-zero off-diagonal elements in **D**. A common method to solve this damped system is to ignore all of the off-diagonal elements of the modal damping matrix. This method is called 'the decoupling approximation'. By applying the method, the system's modal damping matrix is diagonalized to uncouple the system's equation of motion and the Equation (26) is obtained.

Rayleigh's Quotient

We have shown that the system possesses *N* real and positive eigenvalues λ_r and the associated with the real eigenvectors ϕ_r satisfying the eigenvalue problem, Equation (15), $\mathbf{K}\phi_r = \lambda_r \mathbf{M}\phi_r$. The eigenvalues are arranged in ascending order of magnitude, so that they satisfy the inequalities

$$\lambda_1 \le \lambda_2 \le ... \le \lambda_N \tag{27}$$

Equation (12) is premultiplied by ϕ^T,

$$\phi^T \mathbf{K}\phi = \lambda \phi^T \mathbf{M}\phi \tag{28}$$

The positive definiteness of **M** guarantees that $\phi^T \mathbf{M}\phi$ cannot be zero. Therefore,

$$\lambda = \omega^2 = \frac{\phi^T \mathbf{K}\phi}{\phi^T \mathbf{M}\phi} \tag{29}$$

The quotient is called "Rayleigh's quotient" and is a function of eigenvectors ϕ. The behavior of Rayleigh's quotient as ϕ ranges over the entire *N*-dimensional space is of interest. According to the expansion theorem, any arbitrary *N*-vector ϕ can be expressed as a linear combination of the system eigenvectors $\phi_1, \phi_2, ..., \phi_N$.

$$\phi = c_1\phi_1 + c_2\phi_2 + ... + c_N\phi_N \tag{30}$$

$$= \sum_{i=1}^{N} c_i \phi_i \tag{31}$$

$$= \mathbf{\Phi c} \tag{32}$$

where $\mathbf{\Phi}$ is the matrix of eigenvectors of the system and $\mathbf{c} = [c_1 \quad c_2 \quad ... \quad c_N]$ is the *N*-vector of coordinates of ϕ with respect to the basis $\phi_1, \phi_2, ..., \phi_N$. From Equations (29) and (32), we obtain

$$\lambda = \frac{\phi^T \mathbf{K}\phi}{\phi^T \mathbf{M}\phi} = \frac{(\mathbf{\Phi c})^T \mathbf{K}\mathbf{\Phi c}}{(\mathbf{\Phi c})^T \mathbf{M}\mathbf{\Phi c}}$$

$$= \frac{\mathbf{c}^{\mathrm{T}} \mathbf{\Lambda c}}{\mathbf{c}^{\mathbf{T}} \mathbf{c}} = \frac{\sum_{i=1}^{N} \lambda_i c_i^2}{\sum_{i=1}^{N} c_i^2} \tag{33}$$

As the arbitrary vector ϕ moves over the *N*-dimensional Euclidean space, it will eventually enter a small neighborhood of a given eigenvector, say ϕ_r. The coefficients c_i represent the coordinates of ϕ with respect to the bases $\phi_1, \phi_2, ..., \phi_N$. Because ϕ in inside the small neighborhood of ϕ_r, it follows that

$$|c_r| >> |c_i| \qquad i \neq r \tag{34}$$

or

$$\frac{|c_i|}{|c_r|} = \varepsilon_i \qquad i \neq r \tag{35}$$

where ε_i are small numbers. Inserting Equation (35) into Equation (33), using binomial approximation, and ignoring higher-orders terms, we obtain

$$\lambda \cong \lambda_r + \sum_{i=1}^{N} (\lambda_i - \lambda_r) \varepsilon_i^2 \tag{36}$$

But Equation (35) implies that ϕ differs form ϕ_r by a small quantity. Equation (36) states that the corresponding Rayleigh's quotient λ differs form λ_r by a small quantity too. The result says that Rayleigh's quotient corresponding to a linear vibratory system (12) has stationary values in the neighborhood of the eigenvectors, where the stationary values are equal to the associated eigenvalues. If we let $r = 1$ in Equation (36), we will have

$$\lambda = \lambda_1 + \sum_{i=1}^{N} (\lambda_i - \lambda_1) \varepsilon_i^2 \geq \lambda_1 \tag{37}$$

where we recognize that the series is always positive. Inequality (37) states that Rayleigh's quotient is never lower than the lowest eigenvalue λ_1. It is generally higher than λ_1, except when $\phi = \phi_1$, in that case Rayleigh's quotient has a minimum value at $\phi = \phi_1$. The inequality above also gives an upper bound of the lowest eigenvalue λ_1.

Following the similar argument, for $r = N$, Equation (36) yields

$$\lambda = \lambda_N - \sum_{i=1}^{N} (\lambda_N - \lambda_i) \varepsilon_i^2 \leq \lambda_N \tag{38}$$

or Rayleigh's quotient is never higher than the highest eigenvalue λ_N. It is generally lower than λ_N, except when $\phi = \phi_N$, in that case Rayleigh's quotient has a minimum value at $\phi = \phi_N$.

3. MODEL REDUCTION METHODS

These methods are most general techniques computing some approximations to the lower natural frequencies and modes of the undamped large systems $\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$ by solving the symmetric eigenvalue problem in (15).

Selection of Mode Shape Vectors

The mode shape vectors method performance depends on how well the linear combinations of the mode shape vectors $\boldsymbol{\psi}_i$ approximate the natural modes of vibration. There are many approaches to select the approximate mode shape vectors; such as, physical guess of the shapes of the natural modes and a step-by-step computational procedure.

1. Force-Dependent Mode Shape Vectors

Mode shape vectors are determined for analysis of a system with external forces.

$$\mathbf{p}(t) = \mathbf{s}p(t) \tag{39}$$

The spatial distribution of forces **s** does not depend on time; however, the time dependence of the forces is given by the scalar function $p(t)$. The first mode shape vector $\boldsymbol{\psi}_1$ will be the static displacement due to the applied forces **s**, which is

$$\mathbf{K}\mathbf{y}_1 = \mathbf{s} \tag{40}$$

The displacement vector $\mathbf{y}_1$ is normalized to be mass orthonormal:

$$\boldsymbol{\psi}_1 = \frac{\mathbf{y}_1}{(\mathbf{y}_1^T \mathbf{M} \mathbf{y}_1)^{\frac{1}{2}}} \tag{41}$$

The second mode shape vector $\boldsymbol{\psi}_2$ is computed from the static displacement vector $\mathbf{y}_2$ due to the applied forces given by the inertia force distribution associated with the first mode shape vector $\boldsymbol{\psi}_1$. The vector $\mathbf{y}_2$ is obtained from

$$\mathbf{K}\mathbf{y}_2 = \mathbf{M}\boldsymbol{\psi}_1 \tag{42}$$

The second mode shape vector $\boldsymbol{\psi}_2$ is the normalized vector of $\hat{\boldsymbol{\psi}}_2$ where $\hat{\boldsymbol{\psi}}_2$ is created to be orthogonal to, and hence linearly independent of $\boldsymbol{\psi}_1$ by Gram-Schmidt orthogonalization procedure.

The vector $\hat{\boldsymbol{\psi}}_2$ is given by

$$\hat{\boldsymbol{\psi}}_2 = \mathbf{y}_2 - a_{12}\boldsymbol{\psi}_1 \tag{43}$$

and

$$a_{12} = \boldsymbol{\psi}_1^T \mathbf{M} \mathbf{y}_2 \tag{44}$$

Finally the vector $\hat{\boldsymbol{\psi}}_2$ is normalized so that it is mass orthonormal to obtain $\boldsymbol{\psi}_2$.

$$\boldsymbol{\psi}_2 = \frac{\hat{\boldsymbol{\psi}}_2}{(\hat{\boldsymbol{\psi}}_2^T \mathbf{M} \hat{\boldsymbol{\psi}}_2)^{\frac{1}{2}}} \tag{45}$$

The procedure is generalized so that the i^{th} mode shape vector $\boldsymbol{\psi}_i$ is computed form the static displacements y_i due to applied forces given by the inertia force distribution associated with the $(i\text{-}1)^{\text{th}}$ mode shape vector $\boldsymbol{\psi}_{i-1}$. The vector $\mathbf{y}_i$ is determined from

$$\mathbf{K}\mathbf{y}_i = \mathbf{M}\boldsymbol{\psi}_{i-1} \tag{46}$$

The vector $\hat{\boldsymbol{\psi}}_i$ is

$$\hat{\boldsymbol{\psi}}_i = \mathbf{y}_i - \sum_{p=1}^{i-1} a_{pi}\boldsymbol{\psi}_p \tag{47}$$

And the mode shape vector $\boldsymbol{\psi}_i$ is

$$\boldsymbol{\psi}_i = \frac{\hat{\boldsymbol{\psi}}_i}{(\hat{\boldsymbol{\psi}}_i^T \mathbf{M} \hat{\boldsymbol{\psi}}_i)^{\frac{1}{2}}} \tag{48}$$

The series of mode shape vectors $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, ..., \boldsymbol{\psi}_n$ are mutually mass orthonormal and hence they are linear independent of each other. These properties meet the requirement of the mode shape vector method.

2. Modified Force-Dependent Mode Shape Vector Method

Even though, the Gram-Schmidt orthogonalization procedure theoretically gives a new vector that is mass orthogonal to the previous vectors, the actual new vector can suffer loss of orthogonality because of the numerical round-off errors in the computer. To obtain a more stable mode shape vector generation algorithm, an additional set of temporary vectors and orthogonalization procedure are introduced in Chopra [1].

The procedure is modified and summarized as follows:

1. Determine the first mode shape vector $\boldsymbol{\psi}_1$

a.) Determine $\mathbf{y}_1$ by solving: $\mathbf{K}\mathbf{y}_1 = \mathbf{s}$ (49)

b.) Normalize $\mathbf{y}_1$: $\boldsymbol{\psi}_1 = \dfrac{\mathbf{y}_1}{(\mathbf{y}_1^T \mathbf{M} \mathbf{y}_1)^{\frac{1}{2}}}$ (50)

2. Determine additional mode shape vectors $\boldsymbol{\psi}_2, ..., \boldsymbol{\psi}_i, ..., \boldsymbol{\psi}_n$

a.) Determine $\mathbf{y}_i$ by solving: $\mathbf{K}\mathbf{y}_i = \mathbf{M}\boldsymbol{\psi}_{i-1}$ where $i = 2 ... n$ (51)

b.) Orthogonalize $\mathbf{y}_i$ with respect to $\boldsymbol{\psi}_1, ..., \boldsymbol{\psi}_{i-1}$ by repeating the following steps for $p = 1, 2, ..., i-1$

$$a_{pi} = \boldsymbol{\psi}_p^T \mathbf{M} \mathbf{y}_i \quad (52)$$

$$\hat{\boldsymbol{\psi}}_i = \mathbf{y}_i - a_{pi} \boldsymbol{\psi}_p \quad (53)$$

$$y_i = \hat{\boldsymbol{\psi}}_i \quad (54)$$

c.) Normalize $\hat{\boldsymbol{\psi}}_i$: $\boldsymbol{\psi}_i = \dfrac{\hat{\boldsymbol{\psi}}_i}{(\hat{\boldsymbol{\psi}}_i^T \mathbf{M} \hat{\boldsymbol{\psi}}_i)^{\frac{1}{2}}}$ (55)

The force-dependent mode shape vectors methods above use static recurrence procedures to generate mode shape vectors. Therefore, these vectors are best suited only for some low-frequency problems. For high-frequency or banded frequency problems, the force-dependent mode shape vectors methods need large sets of mode shape vectors to span the high-frequency configuration space and thus increase computational cost.

3. Quasi-Static Mode Shape Vector Method

The quasi-static mode shape vector method [2] extends the previous force-dependent mode shape vectors methods by employing a quasi-static recurrence procedure, based on the concept of quasi-static completeness of the mode shape vectors basis. The basic idea is to let the mode shape vectors span the configuration space at desired frequencies and efficiently possess all dynamic deformations for those frequencies. The first quasi-static mode shape vector $\boldsymbol{\psi}_1$ is chosen as a quasi-static mode corresponding to the loading pattern **s** by solving the following equation:

$$(\mathbf{K} - \omega_c^2 \mathbf{M})\mathbf{y}_1 = \mathbf{s} \quad (56)$$

$$\mathbf{y}_1 = (\mathbf{K} - \omega_c^2 \mathbf{M})^{-1} \mathbf{s} \tag{57}$$

where ω_c is the centering frequency [3], which is usually chosen at the midpoint of the frequency range. Normalization of $\mathbf{y}_1$ gives the first quasi-static mode shape vector

$$\boldsymbol{\psi}_1 = \frac{\mathbf{y}_1}{(\mathbf{y}_1^T \mathbf{M} \mathbf{y}_1)^{\frac{1}{2}}} \tag{58}$$

For $i = 2 \ldots n$, the quasi-static recurrence procedure will give additional mode shape vector $\boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_n$

a.) Determine $\mathbf{y}_i$ by solving: $(\mathbf{K} - \omega_c^2 \mathbf{M}) \mathbf{y}_i = \mathbf{M} \boldsymbol{\psi}_{i-1}$ (59)

b.) Orthogonalize $\mathbf{y}_i$

$$\hat{\boldsymbol{\psi}}_i = \mathbf{y}_i - \sum_{j=1}^{i-1} (\boldsymbol{\psi}_j^T \mathbf{M} \mathbf{y}_i) \boldsymbol{\psi}_j \tag{60}$$

c.) Normalize $\hat{\boldsymbol{\psi}}_i$: $\boldsymbol{\psi}_i = \frac{\hat{\boldsymbol{\psi}}_i}{(\hat{\boldsymbol{\psi}}_i^T \mathbf{M} \hat{\boldsymbol{\psi}}_i)^{\frac{1}{2}}}$ (61)

Physically, $\boldsymbol{\psi}_1$ represents a normalized frequency response deformation mode of the undamped system under the loading pattern **s** at the frequency ω_c. By using a quasi-static solution, the dynamic effect of the loading or the inertia term neglected in the static solution of Equations (40) and (49), is included.

4. Mode Shape Vector Termination Procedures

In order to determine how many mode shape vectors are needed for a problem, a participation factor, ρ_i, was introduced by Wilson et al. [6] to measure the significance of one particular mode shape vector, $\boldsymbol{\psi}_i$, in the response

$$\rho_i = \boldsymbol{\psi}_i^T \mathbf{s} \tag{62}$$

The participation factor ρ_i is computed for each mode shape vector, and is used to terminate the vector generation process. The factor ρ_i in Equation (62) does not include the dynamic effects, i.e. it is a static measure; therefore, it is only suitable for low-frequency problems.

Gu et al. proposed a new measure for the participation factor for the system with harmonic external forces:

$$\rho_i = \frac{\left|\boldsymbol{\psi}_i^T \mathbf{r}\right|}{\left[(\boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i)(\mathbf{r}^T \mathbf{M} \mathbf{r})\right]^{\frac{1}{2}}} \tag{63}$$

where

$$\mathbf{r} = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{s} \tag{64}$$

is the frequency response at a specified frequency ω due to the forcing pattern **s**. When the participation factor is one, the mode shape vector perfectly matches the frequency response shape. When the factor is zero, the mode shape vector is orthogonal to the frequency response shape and it does not affect the reduced model response at all. The frequency ω is chosen to be the dominant frequency of the forcing pattern **s**. The participation factor in Equation (63) includes the dynamic effects of the response; hence it is a more realistic factor than the one in Equation (62).

The participation factor is used to terminate the recurrence procedures when the participation vector drops below a specified tolerance value. The sequence of quasi-static recurrence procedures also stops when the number of the mode shape vectors determined is too large.

5. Krylov Subspace

The Krylov subspace is a type of subspace for computing eigenpairs of **B**. This kind of subspace is determined by a nonzero vector **k**. Krylov matrices are $\mathbf{K}^m(\mathbf{k}) = (\mathbf{k}, \mathbf{Bk}, ..., \mathbf{B}^{m-1}\mathbf{k})$ and Krylov subspace $K^m(\mathbf{k}) = \text{span}\left[\mathbf{K}^m(\mathbf{k})\right]$. In principle if all Krylov matrices are saved, they can be used in the Rayleigh Ritz approximations where the Krylov subspace has been computed.

We can see that the model reduction methods, i.e. force-dependent mode shape, the modified force-dependent mode shape and the quasi-static mode shape methods, are based on Krylov subspace iteration method. This is because it forms the Krylov subspace for each mode shape iterated. In order to show that is the case, the eigenvalue problem will be modified to a standard form.

Every real symmetric positive definite matrix **A** can always decomposed into

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T \tag{65}$$

where **L** is a unique nonsingular triangular matrix with positive diagonal elements. Equation (65) is known as the Cholesky Decomposition. We recall that the mass matrix is real symmetric and positive definite. So we can write the mass matrix as follows:

$$\mathbf{M} = \mathbf{L}\mathbf{L}^T \tag{66}$$

For our case where the mass matrix is diagonal, that

$$\mathbf{L} = \mathbf{M}^{\frac{1}{2}} \tag{67}$$

Hence;

$$\mathbf{M} = \mathbf{M}^{\frac{1}{2}}\mathbf{M}^{\frac{1}{2}} \tag{68}$$

The given eigenvalue problem

$$\mathbf{K}\phi_r = \lambda_r \mathbf{M}\phi_r \tag{69}$$

can be rewritten as

$$\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}\phi_r = \lambda_r \mathbf{M}^{\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}\phi_r \tag{70}$$

$$\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}\phi_r = \lambda_r \mathbf{M}^{\frac{1}{2}}\phi_r \tag{71}$$

The equation above can be written in the standard form of the eigenvalue problem

$$\mathbf{A}\mathbf{v}_r = \lambda_r \mathbf{v}_r \tag{72}$$

where

$$\mathbf{A} = \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}} \tag{73}$$

and

$$\mathbf{v}_r = \mathbf{M}^{\frac{1}{2}}\phi_r \tag{74}$$

The first procedure (force-dependent approximate mode shapes) generates the approximate mode shape vectors $\bar{\mathbf{v}}$ from the vector sequence $\mathbf{x}, \mathbf{K}^{-1}\mathbf{M}\mathbf{x}, (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{x}, \ldots$ which is generated in the inverse iteration method. If we multiply $\mathbf{M}^{\frac{1}{2}}$ to $\mathbf{y}_i$ (i = 2,…, n), we will have

$$\bar{\mathbf{y}}_i = \mathbf{M}^{\frac{1}{2}}\mathbf{y}_i \tag{75}$$

Therefore;

$$\mathbf{y}_i = \mathbf{M}^{-\frac{1}{2}}\bar{\mathbf{y}}_i \tag{76}$$

So for the static displacement, we can start with solving for $\bar{\mathbf{y}}_1$

$$\mathbf{K}\bar{\mathbf{y}}_1 = \mathbf{s} \tag{77}$$

The displacement vector $\mathbf{y}_1$ is normalized to be orthonormal:

$$\overline{\mathbf{v}}_1 = \frac{\overline{\mathbf{y}}_1}{(\overline{\mathbf{y}}_1^T \overline{\mathbf{y}}_1)^{\frac{1}{2}}} \tag{78}$$

The second mode shape vector $\overline{\mathbf{v}}_2$ is computed from vector $\overline{\mathbf{y}}_2$ obtained from

$$\overline{\mathbf{y}}_2 = \mathbf{M}^{\frac{1}{2}} \mathbf{K}^{-1} \mathbf{M}^{\frac{1}{2}} \overline{\mathbf{v}}_1 \tag{79}$$

The second mode shape vector $\overline{\mathbf{v}}_2$ is the normalized vector of $\hat{\mathbf{v}}_2$

$$\hat{\mathbf{v}}_2 = \overline{\mathbf{y}}_2 - a_{12} \overline{\mathbf{v}}_1 \tag{80}$$

and

$$a_{12} = \overline{\mathbf{v}}_1^T \mathbf{M} \mathbf{y}_2 \tag{81}$$

Finally the vector $\hat{\mathbf{v}}_2$ is normalized

$$\overline{\mathbf{v}}_2 = \frac{\hat{\mathbf{v}}_2}{(\hat{\mathbf{v}}_2^T \hat{\mathbf{v}}_2)^{\frac{1}{2}}} \tag{82}$$

The vector $\mathbf{y}_i$ is determined from

$$\overline{\mathbf{y}}_i = \mathbf{M}^{\frac{1}{2}} \mathbf{K}^{-1} \mathbf{M}^{\frac{1}{2}} \overline{\mathbf{v}}_{i-1} \tag{83}$$

The vector $\hat{\mathbf{v}}_i$ is

$$\hat{\mathbf{v}}_i = \overline{\mathbf{y}}_i - \sum_{p=1}^{i-1} a_{pi} \overline{\mathbf{v}}_p \tag{84}$$

And vector $\overline{\mathbf{v}}_i$ is

$$\overline{\mathbf{v}}_i = \frac{\hat{\mathbf{v}}_i}{(\hat{\mathbf{v}}_i^T \hat{\mathbf{v}}_i)^{\frac{1}{2}}} \tag{85}$$

It is similar to form a Krylov subspace $\mathbf{K}^m(\mathbf{k}) = (\mathbf{k}, \mathbf{Bk}, ..., \mathbf{B}^{m-1}\mathbf{k})$ with the matrix $\mathbf{B}$ as shown

$$\mathbf{B} = \mathbf{A}^{-1} = \mathbf{M}^{\frac{1}{2}}\mathbf{K}^{-1}\mathbf{M}^{\frac{1}{2}} \tag{86}$$

For the quasi-static vector method, the concept is similar to the above equations shown for force-dependent mode shape method but **B** is to be changed to

$$\mathbf{B} = \mathbf{M}^{\frac{1}{2}}(\mathbf{K} - \omega_c^2\mathbf{M})^{-1}\mathbf{M}^{\frac{1}{2}} \tag{87}$$

6. Subspace Iteration Method

The subspace iteration is another way to find the approximate eigenvectors of the system (12). It is different than the force-dependent mode shape vector, modified force-dependent mode shape vector and quasi-static mode shape vector methods where one approximate mode shape is created at a time but the subspace iteration method carries out iterations to a given number of modes simultaneously [6, pg. 328]. Working with several columns at once will improve the linear convergence of successive subspaces. When several low eigenvalues are clustered, this method will converge to the eigenvectors very fast providing that the initial guess vectors have some directions in the desired eigenvectors.

We propose to make connection between the eigenvalue problem (15) in terms of two real symmetric matrices **M** and **K**. and the standard eigenvalue problem $\mathbf{A}\mathbf{u}_r = \lambda_r\mathbf{u}_r$ in terms of a single real symmetric matrix. By following the Equations (68) through (74), we have

$$\mathbf{A}\mathbf{v}_r = \lambda_r\mathbf{v}_r \tag{72}$$

where

$$\mathbf{A} = \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}} \tag{73}$$

and

$$\mathbf{v}_r = \mathbf{M}^{\frac{1}{2}}\phi_r \tag{74}$$

The mutually orthogonal eigenvectors are assumed to be normalized so as to satisfy $\mathbf{v}_j^T\mathbf{v}_i = \delta_{ij}$.

Subspace iteration is defined by the relation.

$$\hat{\mathbf{V}}_p = \mathbf{A}^{-1}\overline{\mathbf{V}}_{p-1} \qquad \text{where } p = 1,2,\ldots \tag{88}$$

where $\overline{\mathbf{V}}_{p-1}$ is an $N \times n$ matrix of mutually orthonormal vectors $\bar{\mathbf{v}}_i$ related to the matrix $\hat{\mathbf{V}}_{p-1}$ of independent vectors $\hat{\mathbf{v}}_i$ by

$$\overline{\mathbf{V}}_{p-1} = \hat{\mathbf{V}}_{p-1}\mathbf{U}_{p-1} \qquad \text{where } p = 1,2,\ldots \tag{89}$$

where $\mathbf{U}_{p-1}$ is an $n \times n$ upper triangular matrix. Equation (89) expresses the orthonormalization of n independent vectors called the QR factorization. The orthonormalization can be carried by means of the Gram-Schmidt, and this process must be done at every iteration step.

Theorem 1: (QR Factorization) Any $m \times n$ matrix **B** can be written as **B=QR** where **Q** is an $m \times r$ matrix satisfying $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$, **R** is an $r \times n$ upper triangular matrix with nonnegative diagonal elements, $r =$ rank(**B**) and both **Q** and **R** are unique.

The QR factorization is the matrix formulation of the Gram-Schmidt procedure for orthonormalizing the columns of **B** in the order $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$. The set $(\mathbf{q}_1, \mathbf{q}_2, ...\mathbf{q}_j)$ is one orthonormal basis of the subspace spanned by $(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)$

Providing that $\overline{\mathbf{V}}_0$ is not orthogonal to the desired eigenvectors $\mathbf{V}^{(n)}$, the iteration process converges with the result

$$\lim_{p\to\infty} \overline{\mathbf{V}}_p = \mathbf{V}^{(n)} \tag{90}$$

$$\lim_{p\to\infty} \mathbf{U}_p = \mathbf{\Lambda}^{(n)} \tag{91}$$

where $\mathbf{V}^{(n)} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad ... \quad \mathbf{v}_n]$ is the matrix of the n lowest orthonormal eigenvectors and $\mathbf{\Lambda}^{(n)} = \text{diag}[\lambda_1 \quad \lambda_2 \quad ... \quad \lambda_n]$ is the diagonal matrix of the n lowest orthonormal eigenvalues.

The convergence rate of the iteration depends on the gap to the closest eigenvalue not among the n wanted ones. This method is therefore well suited for clustered eigenvalues, where the Power method has a bad convergence rate due to the small gap, and it may sometimes pay to compute a couple of extra vectors to obtain a fast convergence.

7. The Rayleigh-Ritz Procedure

The Rayleigh-Ritz procedure and the theory associated with this method are considered. The Rayleigh-Ritz procedure will improve the accuracy of the approximate mode shape vectors obtained from the Subspace Iteration method previously shown.

Assume that we have a matrix $\overline{\mathbf{V}}$ with orthonormal columns, which are approximations of *n* eigenvectors of **A**. Then further improvement of the estimates can be obtained by using the Rayleigh-Ritz procedure.

Given: $\overline{\mathbf{V}}$, $\overline{\mathbf{V}}^T\overline{\mathbf{V}} = \mathbf{I}$

a. Form $\mathbf{H} := \overline{\mathbf{V}}^T\mathbf{A}\overline{\mathbf{V}}$ (92)

b. Compute the eigenvalues and eigenvectors $\mathbf{H}\mathbf{g}_i = \tau_i\mathbf{g}_i$ and the Ritz vectors $\mathbf{y}_i = \overline{\mathbf{V}}\mathbf{g}_i$ for $i = 1,2,\ldots,n$

c. Compute the residuals $\mathbf{s}_i = \mathbf{A}\mathbf{y}_i - \tau_i\mathbf{y}_i$ for $i = 1,2,\ldots,n$

The matrix **H** is here a $n \times n$ matrix, so it is small compared to **A** if only a few eigenvectors are to be computed. It is also symmetric and positive definite, so Step b can be cheaply computed for the symmetric eigenvalue problem. Hence, the extra work required in this procedure is mainly the work in forming **H**. The eigenvalues $\boldsymbol{\tau}_i$ of **H** and the vectors $\mathbf{y}_i$ are used as new approximations of the eigenvalues and eigenvectors of **A**, respectively. In [7 and 8] Parlett demonstrates three ways in which these Ritz values and Ritz vectors are optimal:

1. The eigenvalues of **A** can be defined by Courant-Fischer Minimax Theorem [10, pg. 411]

$$\lambda_j(\mathbf{A}) = \min_{\mathbf{F}^j \subset C^n} \max_{\mathbf{f} \in \mathbf{F}^j} \frac{\mathbf{f}^*\mathbf{A}\mathbf{f}}{\mathbf{f}^*\mathbf{f}} \qquad \mathbf{f} \neq 0 \text{ and } j = 1, 2,\ldots, n \tag{93}$$

where F^j is a *j* dimensional subspace of C^n. The Ritz values satisfy:

$$\lambda_j(\mathbf{A}) = \min_{\mathbf{G}^j \subset V^n} \max_{\mathbf{g} \in \mathbf{G}^j} \frac{\mathbf{g}^T\mathbf{A}\mathbf{g}}{\mathbf{g}^T\mathbf{g}} \qquad \mathbf{g} \neq 0 \text{ and } j = 1, 2,\ldots, n \tag{94}$$

where $\overline{\mathbf{V}}^n = \text{span}(\overline{\mathbf{V}})$ and $\mathbf{G}^j$ is a *j* dimensional subspace of $\overline{\mathbf{V}}^n$. This is a natural definition of the best approximation to $\lambda_j(\mathbf{A})$ in the subspace $\overline{\mathbf{V}}^n$.

2. Define the residual matrix

$$\mathbf{R(B)} = \mathbf{A}\overline{\mathbf{V}} - \overline{\mathbf{V}}\mathbf{B}. \tag{95}$$

Then the matrix $\mathbf{H} := \overline{\mathbf{V}}^T\mathbf{A}\overline{\mathbf{V}}$ minimizes this residual, i.e. $\|\mathbf{R(H)}\| < \|\mathbf{R(B)}\|$ if $\mathbf{B} \neq \mathbf{H}$.

3. The Ritz pairs are the eigenpairs for **A**'s projection onto $\overline{\mathbf{V}}^n$, i.e. the matrix which spans the closest subspace in $\overline{\mathbf{V}}^n$ to span(**A**).

The minimum value of the norm of the residual matrix **R(B**) can be seen as a measure of how far $\overline{\mathbf{V}}^n$ is from being an invariant subspace of **A**. If $\overline{\mathbf{V}}^n$ is an invariant subspace of **A** then the product $\mathbf{A}\bar{\mathbf{v}}_i$, where $\bar{\mathbf{v}}_i$ is a column of $\overline{\mathbf{V}}$, equals a linear combination of the columns of $\overline{\mathbf{V}}$, i.e. $\mathbf{A}\bar{\mathbf{v}}_i = \overline{\mathbf{V}}\mathbf{b}$. Hence, there is a matrix **B** such that **R(B)** is zero. If $\overline{\mathbf{V}}$ is orthogonal then $\mathbf{B} = \overline{\mathbf{V}}^T\mathbf{A}\overline{\mathbf{V}} =: \mathbf{H}$ and **H** is the restriction of **A** to $\overline{\mathbf{V}}^n$. If $\overline{\mathbf{V}}^n$ is not an invariant subspace of **A**, then there is no matrix **B** such that $\|\mathbf{R(B)}\| = 0$,but the matrix $\mathbf{H} = \overline{\mathbf{V}}^T\mathbf{A}\overline{\mathbf{V}}$ still minimizes **R(B)**.

Moreover, if $\mathbf{W} = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad ... \quad \mathbf{w}_n]$ is any orthonormal basis for $\overline{\mathbf{V}}^n$ and $\mathbf{D} = \text{diag}(d_1, d_2, ..., d_n)$ is any diagonal matrix, then $\|\mathbf{AW} - \mathbf{WD}\|$ is minimized when and only when $\mathbf{w}_i = \mathbf{y}_i$ and $d_i = \tau_i$ for i = 1, 2,…, n. This follows since, if $\mathbf{w}_i = \mathbf{y}_i$ and $d_i = \tau_i$ for i = 1, 2,…, n, then

$$\|\mathbf{A\Psi} - \mathbf{\Psi T}\| = \|\mathbf{A}\overline{\mathbf{V}}\mathbf{G} - \overline{\mathbf{V}}\mathbf{GT}\| \tag{96}$$

$$= \|\mathbf{A}\overline{\mathbf{V}} - \overline{\mathbf{V}}\mathbf{GTG}^T\| \tag{97}$$

$$= \|\mathbf{AV} - \mathbf{VH}\| \tag{98}$$

$$= \|\mathbf{R(H)}\| \tag{99}$$

Here we have used the notations $\mathbf{Y} = [\mathbf{y}_1 \quad \mathbf{y}_2 \quad ... \quad \mathbf{y}_n]$, $\mathbf{T} = \text{diag}(\tau_1, \tau_2, ...\tau_n)$ and $\mathbf{G} = \text{diag}(\mathbf{g}_1, \mathbf{g}_2, ...\mathbf{g}_n)$.

But when $\mathbf{W} \neq \mathbf{Y}$ or $\mathbf{N} \neq \mathbf{T}$, we can still express **W** in the basis $\overline{\mathbf{V}}$, $\mathbf{W} = \overline{\mathbf{V}}\mathbf{N}$, $\mathbf{N}^T\mathbf{N} = \mathbf{I}$, since they span the same subspace. We obtain

$$\|\mathbf{AW} - \mathbf{WD}\| = \|\mathbf{A}\overline{\mathbf{V}}\mathbf{N} - \overline{\mathbf{V}}\mathbf{ND}\| \tag{100}$$

$$= \|\mathbf{A}\overline{\mathbf{V}} - \overline{\mathbf{V}}\mathbf{NDN}^T\| \tag{101}$$

$$> \|\mathbf{R(H)}\| \tag{102}$$

This means that when $\overline{\mathbf{V}}^n$ is an invariant subspace of **A** the Ritz pairs are the true eigenpairs of **A**.

When the eigenvalues of matrix **A** are well separated there are easily obtained bounds for the Ritz values once the residuals $\mathbf{s}_i$ have been computed. These bounds are given in the theorem.

Theorem 2: (residual error bound) Let $\overline{\mathbf{V}} \in \mathbf{C}^{N\times n}$ be a matrix satisfying $\overline{\mathbf{V}}^*\overline{\mathbf{V}} = \mathbf{I}$ and let $(\tau_i, \mathbf{y}_i)$ for i = 1, 2,…, n be the corresponding set of Ritz pairs for **A** with residuals $\mathbf{s}_i = \mathbf{A}\mathbf{y}_i - \tau_i\mathbf{y}_i$. Then the interval $\left[\tau_i - \|\mathbf{s}_i\|, \tau_i + \|\mathbf{s}_i\|\right]$ contains an eigenvalue of **A**.

This theorem, together with the following theorem, can also be found in [7 and 8, Chapt. 11].

Proof:

Let $\hat{\lambda}$ be the closest eigenvalue of **A** to τ_i. If $\hat{\lambda} = \tau_i$ the result is immediate. If $\hat{\lambda} \neq \tau_i$ then the matrix $\mathbf{A} - \tau_i\mathbf{I}$ is non-singular. Using $\mathbf{y}_i = (\mathbf{A} - \tau_i\mathbf{I})^{-1}(\mathbf{A} - \tau_i\mathbf{I})\mathbf{y}_i$ gives

$$1 = \|\mathbf{y}_i\| \le \left\|(\mathbf{A} - \tau_i\mathbf{I})^{-1}\right\| \|(\mathbf{A} - \tau_i\mathbf{I})\mathbf{y}_i\| \tag{103}$$

$$= \left(\frac{1}{\min_j |\lambda_j(\mathbf{A}) - \tau_i|}\right) \cdot \|\mathbf{s}_i\| \tag{104}$$

Hence, $\hat{\lambda}$ satisfies

$$\left|\hat{\lambda} - \tau_i\right| \le \|\mathbf{s}_i\| \tag{105}$$

which proves the theorem.

If all the intervals corresponding to the Ritz values are disjoint we know that inside each of these intervals there is an eigenvalue of **A**. Thus, we have n approximate eigenvalues of known accuracy. However, if some of the intervals overlap there may be two Ritz values approximating the same eigenvalue. An additional bound, for the Ritz values in overlapping intervals as well, is given in Theorem below.

Theorem 3: Let $\overline{\mathbf{V}} \in \mathbf{C}^{N\times n}$ be a matrix satisfying $\overline{\mathbf{V}}^*\overline{\mathbf{V}} = \mathbf{I}$ and let $(\tau_i, \mathbf{y}_i)$ for i = 1, 2,…, n be the corresponding set of Ritz pairs for **A** with residuals $\mathbf{s}_i = \mathbf{A}\mathbf{y}_i - \tau_i\mathbf{y}_i$. Then there are n eigenvalues of **A**, λ_j, j = 1, 2,…, n such that

$$\left|\tau_i - \lambda_j\right| \le \|\mathbf{S}\| \tag{106}$$

where

$$\mathbf{S} = \begin{bmatrix}\mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_n\end{bmatrix} = \mathbf{AY} - \mathbf{YT} \tag{107}$$

Proof:

It is always possible to find a unitary matrix $\tilde{\mathbf{V}}$, so the matrix $\mathbf{P} = \begin{bmatrix}\overline{\mathbf{V}} & \tilde{\mathbf{V}}\end{bmatrix}$ is square and $\mathbf{P}^*\mathbf{P} = \mathbf{I}$. Then multiplying matrix **A** from left and right by $\mathbf{P}^*$ and **P**, respectively, gives

$$\mathbf{P}^*\mathbf{AP} = \begin{bmatrix}\overline{\mathbf{V}}^*\mathbf{A}\overline{\mathbf{V}} & \overline{\mathbf{V}}^*\mathbf{A}\tilde{\mathbf{V}} \\ \tilde{\mathbf{V}}^*\mathbf{A}\overline{\mathbf{V}} & \tilde{\mathbf{V}}^*\mathbf{A}\tilde{\mathbf{V}}\end{bmatrix} := \begin{bmatrix}\mathbf{H} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{Q}\end{bmatrix} \tag{108}$$

Let **R=R(H)=A** $\overline{\mathbf{V}}$ **-** $\overline{\mathbf{V}}$ **H**. Then

$$\mathbf{P}^*\mathbf{R} = \mathbf{P}^*\mathbf{A}\overline{\mathbf{V}} - \mathbf{P}^*\overline{\mathbf{V}}\mathbf{H} \tag{109}$$

$$= (\mathbf{P}^*\mathbf{AP})(\mathbf{P}^*\overline{\mathbf{V}}) - (\mathbf{P}^*\overline{\mathbf{V}})\mathbf{H} \tag{110}$$

$$= \begin{bmatrix}\mathbf{H} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{Q}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}\mathbf{I} \\ \mathbf{0}\end{bmatrix}\mathbf{H} \tag{111}$$

$$= \begin{bmatrix}\mathbf{0} \\ \mathbf{F}\end{bmatrix} \tag{112}$$

so $$\|\mathbf{R}\| = \left\|\mathbf{P}^*\mathbf{R}\right\| = \|\mathbf{F}\| \tag{113}$$

Split the matrix $\mathbf{P}^*\mathbf{AP}$ into

$$\mathbf{P}^*\mathbf{AP} = \begin{bmatrix}\mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{0}\end{bmatrix} := \tilde{\mathbf{Q}} + \tilde{\mathbf{F}} \tag{114}$$

Then by the Weyl Monotonicity Theorem [7, pg. 192] the eigenvalues of $\mathbf{P}^*\mathbf{AP}$ satisfy

$$\lambda_i(\mathbf{A}) = \lambda_i(\mathbf{P}^*\mathbf{AP}) \le \lambda_i(\tilde{\mathbf{Q}}) + \lambda_i(\tilde{\mathbf{F}}) \tag{115}$$

Now, since the eigenvalues of $\tilde{\mathbf{Q}}$ are the union of the eigenvalues of **H** and **Q**, each Ritz value τ_i equals an eigenvalue of $\tilde{\mathbf{Q}}$. Hence, there are indices *j* such that

$$\lambda_j(\tilde{\mathbf{Q}}) = \tau_i \qquad \text{for } i = 1, 2, \ldots, n \tag{116}$$

The second term in Equation (115) is obtained by computing

$$\tilde{\mathbf{F}}^2 = \begin{bmatrix} \mathbf{0} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^*\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\mathbf{F}^* \end{bmatrix} \tag{117}$$

Since $\mathbf{F}^*\mathbf{F}$ and $\mathbf{F}\mathbf{F}^*$ have the same eigenvalues we get

$$\lambda_1(\tilde{\mathbf{F}}) = \sqrt{\lambda_1(\mathbf{F}^*\mathbf{F})} = \sqrt{\|\mathbf{F}\|^2} = \|\mathbf{R}\| \tag{118}$$

With $\|\mathbf{R}\| = \|\mathbf{S}\|$, in Equations (115) and (116) when *i=j* we have

$$\lambda_j(\mathbf{A}) - \tau_i \le \|\mathbf{S}\| \tag{119}$$

Similarly,

$$\lambda_i(\tilde{\mathbf{Q}}) \le \lambda_i(\mathbf{P}^*\mathbf{A}\mathbf{P}) + \lambda_1(-\tilde{\mathbf{F}}) \tag{120}$$

and, recalling the eigenvalue distribution of the matrix $\tilde{\mathbf{F}}$. We know that $\lambda_1(-\tilde{\mathbf{F}}) = \|\mathbf{S}\|$. Hence, by the same arguments as above we have

$$\tau_i - \lambda_j(\mathbf{A}) \le \|\mathbf{S}\| \tag{121}$$

and the inequality (106) is established.

When considering the accuracy of the Ritz vectors the problem is not as simple as it is for the Ritz values. The reason is that eigenvectors associated with multiple eigenvalues are not uniquely determined. Any linear combination of eigenvectors corresponding to the same eigenvalue is an eigenvector. Similarly, Ritz vectors corresponding to eigenvalues which are close tend to be very sensitive and give bad estimates to the corresponding eigenvectors, but the subspace these vectors span may be a good approximate of the subspace associated with the cluster. A bound for how well a Ritz vector approximate an eigenvector of **A** does exist, but it is only useful when the associated eigenvalues are well separated.

Theorem 4: (residual bounds) Let $\overline{\mathbf{V}} \in \mathbf{C}^{N\times n}$ be a matrix satisfying $\overline{\mathbf{V}}^*\overline{\mathbf{V}} = \mathbf{I}$ and let $(\tau_i, \mathbf{y}_i)$ for i = 1, 2,…, n be the corresponding set of Ritz pairs for **A** with residuals $\mathbf{s}_i = \mathbf{A}\mathbf{y}_i - \tau_i\mathbf{y}_i$. Then the eigenvalues of **A**, λ_i, i = 1, 2,…, n satisfy

$$\left|\tau_i - \lambda_i\right| \le \frac{\|\mathbf{S}\|^2}{Gap} \tag{122}$$

where

$$\mathbf{S} = \begin{bmatrix}\mathbf{s}_1 & \mathbf{s}_2 & \ldots & \mathbf{s}_n\end{bmatrix} = \mathbf{AY} - \mathbf{YT} \tag{123}$$

and "Gap" is the gap between the maximum of the eigenvalues of **A** and the minimum of those of **F**.

Proof:

A is similar to $\mathbf{P}^*\mathbf{AP}$ in Equation (108)

$$\mathbf{P}^*\mathbf{AP} = \begin{bmatrix}\overline{\mathbf{V}}^*\mathbf{A}\overline{\mathbf{V}} & \overline{\mathbf{V}}^*\mathbf{A}\widetilde{\mathbf{V}} \\ \widetilde{\mathbf{V}}^*\mathbf{A}\overline{\mathbf{V}} & \widetilde{\mathbf{V}}^*\mathbf{A}\widetilde{\mathbf{V}}\end{bmatrix} \coloneqq \begin{bmatrix}\mathbf{H} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{Q}\end{bmatrix} \tag{124}$$

Take the determinant of $(\lambda\mathbf{I} - \mathbf{A})$, one has

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix}\lambda\mathbf{I} - \mathbf{H} & -\mathbf{F}^* \\ -\mathbf{F} & \lambda\mathbf{I} - \mathbf{Q}\end{bmatrix}\right) \tag{125}$$

$$= \det(\lambda\mathbf{I} - \mathbf{Q}) \cdot \det\left[\lambda\mathbf{I} - \mathbf{H} - \mathbf{F}^*(\lambda\mathbf{I} - \mathbf{Q})^{-1}\mathbf{F}\right] \tag{126}$$

The eigenvalues of **A** near the eigenvalue of **H** are the eigenvalues of $\mathbf{H} + \mathbf{F}^*(\lambda\mathbf{I} - \mathbf{Q})\mathbf{F}$, therefore

$$\left|\tau_i - \lambda_i\right| \le \frac{\|\mathbf{F}\|^2}{Gap} \tag{127}$$

$$= \frac{\|\mathbf{S}\|^2}{Gap} \tag{128}$$

Some techniques e.g. previously described subspace iteration produce the approximate eigenvectors that are not mutually orthogonal when the eigenvalues are close together. This also happens when n is large. Let $\overline{\mathbf{V}}$ be an $N \times n$ matrix with

orthonormal columns, which are approximations of *n* eigenvectors of **A,** probably the matrix of Ritz vectors. When the smallest singular value of $\overline{\mathbf{V}}$ is 1 i.e. $\sigma_{\min}(\overline{\mathbf{V}}) = 1$, then $\sigma_{maz}(\overline{\mathbf{V}}) = 1$ too. Thus $\overline{\mathbf{V}}$ is orthonormal. Theorem 3 will be weak if $\sigma_{\min}(\overline{\mathbf{V}})$ decreases and gets close to 0. Theorem 3 would be $|\tau_i - \lambda_i| \le \frac{\|\mathbf{S}\|}{\sigma_{\min}(\overline{\mathbf{V}})}$ when $\overline{\mathbf{V}}$ is not orthonormal.

Suppose we have a matrix $\overline{\mathbf{V}}$, a set of Ritz pairs for **A** i.e. $(\tau_i, \mathbf{y}_i)$ and the residuals $\mathbf{s}_i = \mathbf{A}\mathbf{y}_i - \tau_i \mathbf{y}_i$. For each *i* = 1, 2,…, *n* there is an eigenvalue λ of **A** such that $|\lambda - \tau_i| \le \|\mathbf{s}_i\|$. But there may not be *n* distinct λ's for each *i.* Therefore we need the smallest η such that a distinct λ may be found in each interval $[\tau_i - \eta, \tau_i + \eta]$ and $\eta = \frac{\|\mathbf{S}\|}{\sigma_{\min}(\overline{\mathbf{V}})}$.

When $\overline{\mathbf{V}}$ is not quite orthonormal, theorem 3 has to be modified as follows.

Theorem 5: Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ hold the eigenvalues of **A** and $\tau = (\tau_1, \tau_2, ..., \tau_n)$ hold the eigenvalues of **H**. And let $\overline{\mathbf{V}}$ be a $N \times n$ matrix with full column rank. Then there are at least *n* locations $\lambda_1, \lambda_2, ...\lambda_j, ..., \lambda_n$ in λ such that, for *j* = 1, 2,…, *n*

$$|\tau_i - \lambda_j| \le \frac{\|\mathbf{S}\|}{\sigma_{\min}(\overline{\mathbf{V}})} \tag{129}$$

where

$$\mathbf{S} = [\mathbf{s}_1 \quad \mathbf{s}_2 \quad ... \quad \mathbf{s}_n] = \mathbf{AY} - \mathbf{YT} \tag{130}$$

Please see [8, pg. 258] for the proof.

8. Reduced Equations of Motion

From Equation (11), the equation of motion for a system with *N* degrees of freedom subjected to a force $\mathbf{p}(t) = \mathbf{s}p(t)$ is

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{s}p(t) \tag{131}$$

In the mode shape vector methods, the displacements are expressed as a linear combination of the several shape vectors $\boldsymbol{\psi}_j$, that is

$$\mathbf{x}(t) = \sum_{j=1}^{n} z_j(t)\boldsymbol{\psi}_j = \boldsymbol{\Psi}\mathbf{z}(t) \tag{132}$$

where $z_j(t)$ are generalized coordinates, $\boldsymbol{\psi}_j$ are approximate mode shape vectors computed from the previously explained model reduction methods and $\boldsymbol{\Psi}$ is the matrix whose columns are $\boldsymbol{\psi}_j$. These approximate mode shape vectors are linearly independent vectors satisfying the geometric boundary conditions.

However if the subspace iteration method or the Rayleigh Ritz methods are used, introduce the transition matrix

$$\boldsymbol{\Psi} = \mathbf{M}^{-\frac{1}{2}}\overline{\mathbf{V}} \tag{133}$$

and

$$\mathbf{x}(t) = \boldsymbol{\Psi}\mathbf{z}(t) \tag{134}$$

Hence, substituting the approximate mode shape vectors in Equations (132) and (134) into the equation of motion will result in

$$\mathbf{M}\boldsymbol{\Psi}\ddot{\mathbf{z}}(t) + \mathbf{C}\boldsymbol{\Psi}\dot{\mathbf{z}}(t) + \mathbf{K}\boldsymbol{\Psi}\mathbf{z}(t) = \mathbf{s}p(t) \tag{135}$$

Premultiply by $\boldsymbol{\Psi}^{\mathrm{T}}$ gives

$$\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Psi}\ddot{\mathbf{z}}(t) + \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Psi}\dot{\mathbf{z}}(t) + \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Psi}\mathbf{z}(t) = \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{s}p(t) \tag{136}$$

$$\mathbf{I}\ddot{\mathbf{z}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{z}}(t) + \tilde{\mathbf{K}}\mathbf{z} = \tilde{\mathbf{L}}p(t) \tag{137}$$

where

$$\begin{aligned} \tilde{\mathbf{C}} &= \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Psi} \\ \tilde{\mathbf{K}} &= \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Psi} \\ \tilde{\mathbf{L}} &= \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{s} \end{aligned} \tag{138}$$

Equation (137) is a system of *n* differential equations in the *n* generalized coordinates **z**(*t*). The coordinate transformation of equation (134) can reduce the original set of *N* equations (133) into the nodal displacement **x** to a smaller set of *n* equations in the generalized coordinates **z**. It is questionable that only a few mode shape vectors (*n* is much smaller than *N*) would be good enough to represent the displacements of the system. Also the selection of the mode shape vectors is critical.

With Equation (137), we can find the eigenvectors of the reduced model $\mathbf{z}_i$ with less work because the degrees of freedom of this equation of motion were reduced from *N* to *n* and *n* is small.

Approximation

In the analysis of Equation (137), the approximate natural frequencies are from the square root of the approximate eigenvalues obtained from the model reduction methods.

$$\tilde{\omega}_i = \sqrt{\lambda_i} \tag{139}$$

Moreover, they cannot be smaller than the actual frequencies according to the Rayleigh's stationarity condition

$$\omega_1 \le \tilde{\omega}_1, \ \omega_2 \le \tilde{\omega}_2, \ldots, \ \omega_n \le \tilde{\omega}_n \tag{140}$$

One can see that the damping and the stiffness matrices in Equation (137) are not diagonal. If the off-diagonal elements of the damping matrix $\tilde{\mathbf{C}}$ and stiffness matrix $\tilde{\mathbf{K}}$ are small comparing to the diagonal elements, there is a common method to solve such a system that is to ignore all of the off-diagonal elements and keep only the diagonal elements. Then one can solve the uncoupled differential equations.

$$\ddot{z}_i(t) + \tilde{c}_i \dot{z}_i(t) + \tilde{\omega}_i^2 z_i(t) = \tilde{L}_i p(t) \qquad i = 1, 2, \ldots, n \tag{141}$$

4. CONCLUSION

This report describes a structural dynamic analysis by using approximate mode shape vectors in the large systems to obtain some mode shapes of the small systems and solving for the responses based on those small systems. A small system always needs less computation cost and time to compute the natural frequencies, mode shapes and responses. Most of the time a small system obtained serves really well in providing a good analysis of a structure. There are force-dependent mode shape, modified force-dependent mode shape, quasi-static mode shape, subspace iteration and Rayleigh Ritz methods described in this report. The force-dependent mode shape vector method computes the first mode shape vector based on the static response mode, but the inertia term is neglected. Then the inertia is applied as a static load in the next step to generate a new mode shape vector. This process is repeated till there are enough mode shape vectors which satisfy static completeness condition. The algorithm loses orthogonality due to the numerical round-off errors when the number of the mode shape vectors becomes large. The force-dependent mode shape vector method is then modified with an additional set of temporary vectors and a new Gram-Schmidt orthogonalization procedure to get more stable mode shape vectors. However, when the mode shape vector set from the force-dependent mode shape vector method becomes very large, the mode shape vectors become nearly linearly dependent, causing loss of accuracy. In order to obtain more accurate results, the quasi-static mode shape vector method is preferred since the

dynamic effect of the loading or the inertia term is included in the quasi-static solution. However, with subspace iteration method, specific number of mode shapes required are approximated once every iteration. Rayleigh Ritz shown in section 7 requires some extra work; but the Ritz vectors are more accurate than the approximate mode shapes from the other methods explained in this report. Nevertheless; there is another good option called the Lanczos algorithm but it is not covered in this report. Lanczos algorithm implements the Rayleigh Ritz procedure on the sequence of Krylov subspace where the Rayleigh Ritz procedure is simplified.

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