An Ordinary Differential Equation in Interval-Arithmetic

Let points u in the plane be employed as the diagonally opposite corners of a rectangular *interval* $[\underline{u}, \overline{u}]$ when $\underline{u} \le \overline{u}$ componentwise. In other words, point $u = (x, y)$ belongs to interval $[\underline{u}, \overline{u}]$ with $\underline{u} = (\underline{x}, \underline{y})$ and $\overline{u} = (\overline{x}, \overline{y})$ just when respective components satisfy $\underline{x} \le x \le \overline{x}$ and $\underline{y} \le y \le \overline{y}$. Such rectangular intervals, with edges parallel to coordinate axes, shall be called "coffins"; their restricted orientation renders them disadvantageous for the purpose of bounding the propagation of uncertainty into the solutions of most differential equations, as this note will illustrate by an example.

The differential equation of uniform counter-clockwise rotary motion of the plane about its origin is $du/dt = u \cdot J$ in which $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. This rotates any point u at time $t = 0$ to $u \cdot \exp(tJ)$ at time t , which is also the angle of rotation; in fact, $\exp(tJ) = I \cdot \cos(t) + J \cdot \sin(t)$. Rectangular intervals should get rotated the same way except that then their edges would not always lie parallel to the coordinate axes. If we insist that point-sets rotated by the differential equation be circumscribed always by coffins of the kind in the first paragraph above, the coffins will grow too fast. How much too fast?

Suppose that the initial point u is uncertain to the extent that only a coffin $[\underline{u}, \overline{u}]$ containing it is known. The differential equation moves points u that lie initially in an interval $[\underline{u}, \overline{u}]$ into a circumscribing coffin whose corners can be seen to satisfy

$$d\underline{u}/dt = \min\{u \cdot J\} \quad \text{and} \quad d\overline{u}/dt = \max\{u \cdot J\} \quad \text{min/maximized over } u \text{ in } [\underline{u}, \overline{u}] \,.$$

Spelled out in detail, these equations say

$$d\underline{x}/dt = \min\{-y\} = -\overline{y} \,, \quad d\underline{y}/dt = \min\{x\} = \underline{x} \,, \quad d\overline{x}/dt = \max\{-y\} = -\underline{y} \,, \quad d\overline{y}/dt = \max\{x\} = \overline{x} \,.$$

The size of the coffin $[\underline{u}, \overline{u}]$ is determined by $\overline{u}-\underline{u}$, whose components satisfy

$$d(\overline{x}-\underline{x})/dt = (\overline{y}-\underline{y}) \quad \text{and} \quad d(\overline{y}-\underline{y})/dt = (\overline{x}-\underline{x}) \,; \quad \text{i.e.,} \quad d(\overline{u}-\underline{u})/dt = (\overline{u}-\underline{u}) \cdot |J| \ \text{ elementwise.}$$

The last differential equation's solution transforms $(\overline{u}-\underline{u})$ at time $t = 0$ to $(\overline{u}-\underline{u}) \cdot \exp(t|J|)$ in which $\exp(t|J|) = I \cdot \cosh(t) + |J| \cdot \sinh(t)$ grows exponentially. Thus, although the differential equation is not uncertain at all, its effect upon its initial point's uncertainty is changed from mere rotation to explosive exponential growth by the insistence that uncertainty be bounded by coffins instead of rectangles with arbitrary orientations. Adding uncertainty to the differential equation can only exacerbate the growth.

This example is typical of the way naive interval arithmetic, when used to bound the propagation of uncertainty, can cause it to grow exponentially too fast. The exponentially excessive growth can be somewhat damped sometimes by allowing parallelograms instead of coffins, and general skew parallelepipeds instead of coffin-shaped rectangular parallelepipeds in higher-dimensional spaces. The suppression of exponentially excessive growth generally requires relaxation of the restrictions upon shape as well as orientation. Among methods known so far, growth is least excessive, typically less than linear, when uncertainty is bounded by ellipsoids rather than by parallelepipeds, but that's a story for another day; see
`http://www.cs.berkeley.edu/~wkahan/Math128/Ellipsoi.pdf`