M. Brand's 2nd Revised Problem has Infinitely Many Solutions

Given are two real symmetric positive definite matrices $A = A'$ and $V = V'$, and real column b and row c' both nonzero. Desired are the real solutions Z of the equation

$$Z'AZ + bc'Z - V = O\ .$$

Note that Z may be rectangular provided it has at least as many rows (same number as A's) as columns (same number as V's).

The *Choleski Factorizations* of $A = U'U$ and $V = R'R$ provide two upper-triangular matrices U and R. Set $Z := U^{-1}SR$, $p := R'^{-1}b \neq o$ and $g' := c'U^{-1} \neq o'$ to transform the given equation for Z into an equivalent equation we wish to solve for S :

$$S'S + pg'S - I = O\ .$$

Let $e' := [1\ 0\ 0\ \dots\ 0]$ be the first row of any identity matrix of appropriate dimensions. Next compute symmetric orthogonal matrices $W = W' = W^{-1}$ and $Y = Y' = Y^{-1}$ that map p to $Wp = \pm\|p\|e$ and g' to $g'Y = \pm\|g\|e'$ wherein $\|p\| = \sqrt{(p'p)}$ and $\|g\| = \sqrt{(g'g)}$; compute $W := 2ww'/w'w - I$ from $w := p \pm \|p\|e \neq o$, and $Y := 2yy'/y'y - I$ from $y := g \pm \|g\|e \neq o$. Now $Wpg'Y = \mu ee'$ for a scalar $\mu := \pm\|p\|\cdot\|g\|$ whose sign is inherited from earlier choices of $\pm$ signs. Consequently we shall obtain a desired solution $S := YKW$ from any solution K of

$$K'K + \mu ee'K - I = O\ .$$

Every such solution $K = \begin{bmatrix} \beta & o' \\ f & F \end{bmatrix}$; here F has orthonormal columns (i.e. $F'F = I$) but is otherwise arbitrary, f is any solution of $F'f = o$ with $f'f \le 1 + \mu^2/4$, and $ß := -\mu/2 \pm \sqrt{(1 - f'f + \mu^2/4)}$ has either of two values. (The other is $-(1-f'f)/ß$.) Note that F, like Z, must have at least as many rows as columns; but if F and Z are square matrices then $F' = F^{-1}$ and $f = o$.

Thus, every solution Z can be computed by first computing U and R, then p, g' and μ. Then choose any F of the right dimensions with orthonormal columns and, if it is not square, choose any f orthogonal to F's columns and not too big: $f'f \le 1 + \mu^2/4$. Then choose one of two available values for $ß$ and construct K, Y, W, S and finally Z.

If Z is square (so $f = o$) two extremal solutions can be identified: one has $F = -I$ and $ß < 0$; the other has $F = I$ and $ß > 0$. The two values of $ß$ are the values of

$$-(\ \mu \pm \sqrt{(4 + \mu^2)}\ )/2 \quad \text{and} \quad 2/(\ \mu \pm \sqrt{(4 + \mu^2)}\ )\ ;$$ choose $\pm$ signs to avoid cancellation.

Here is a MATLAB program to provide both extremal solutions in this square case:

```
function  [Zhi,  Zlo] = brand(A, b, c, V)
b = b(:) ;  c = c(:) ;  U = chol(A) ;  R = chol(V) ;  p = R’\b ;  g = U’\c ;
lp = norm(p) ;  lg = norm(g) ;  mu = lp*lg ;  w = p ;  y = g ;
if  w(1)>0 ,  w(1) = w(1) + lp ;  else  w(1) = w(1) - lp ;  mu = -mu ;  end
if  y(1)>0 ,  y(1) = y(1) + lg ;  else  y(1) = y(1) - lg ;  mu = -mu ;  end
k = ones(length(b)-1,1) ;  d = sqrt(mu*mu + 4) + abs(mu) ;
if  mu>0 ,  blo = -d/2 ;  bhi = 2/d ;  else  blo = -2/d ;  bhi = d/2 ;  end
Klo = [blo; -k] ;  Khi = [bhi; k] ;  cw = 2/(w’*w) ;  cy = 2/(y’*y) ;
S = (cw*(Klo.*w))*w’ - diag(Klo) ;  Slo = (cy*y)*(y’*S) - S ;
S = (cw*(Khi.*w))*w’ - diag(Khi) ;  Shi = (cy*y)*(y’*S) - S ;
Zlo = U\Slo*R ;  Zhi = U\Shi*R ;
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This program was tested on randomly generated examples of which some samples follow:

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$$A = \begin{bmatrix} 164 & -1 & -58 \\ -1 & 229 & -57 \\ -58 & -57 & 174 \end{bmatrix}, \quad b = \begin{bmatrix} -10 \\ 5 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 9 \\ -1 \\ -2 \end{bmatrix}, \quad V = \begin{bmatrix} 85 & -17 & 66 \\ -17 & 255 & 5 \\ 66 & 5 & 334 \end{bmatrix};$$

Zhi =

1.109840049881680	-0.4253931355746230	0.5001629792830848
0.1206715574191866	1.011046773463334	0.5048722534871607
0.3828893646171346	-0.1442649562470549	1.543191771916220

Zlo =

-0.5371541152388652	0.1390501682532156	-0.4428943858188037
-0.1449099635028275	-0.9989275704215140	-0.5072960940955248
-0.3148767493153201	0.1102586485961477	-1.536390510386038

$$A = \begin{bmatrix} 85 & -58 & -13 \\ -58 & 105 & 8 \\ -13 & 8 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1108 \\ 19 \\ -1307 \end{bmatrix}, \quad c = \begin{bmatrix} 64 \\ 423 \\ -16 \end{bmatrix}, \quad V = \begin{bmatrix} 85 & 27 & -81 \\ 27 & 106 & 46 \\ -81 & 46 & 131 \end{bmatrix};$$

Zhi =

5.167098224202696	0.1607300268144672	6.054872185497971
0.4813216129673538	0.2477840131363260	0.6024937430376918
33.38889876937286	7.193633915618992	40.15314775155887

Zlo =

-4437.167098224607	-76.16073002682194	5221.945127814978
-7756.481321613002	-133.2477840131375	9148.397506257003
11046.61110122815	182.8063660843385	-13110.15314774864

How accurate these results are is hard to say. However, their residuals have been computed and are comparable with the rounding errors generated when they were computed, so these numerical results are compatible with a claim of "Backward Stability". In other words, the computed Z's cannot be much worse than if they had been computed exactly from data perturbed only in end-figures.

The foregoing formulas must suffer excessively from roundoff only when A and V are too nearly singular. This situation may arise artificially when A and V were derived from other data in a way that happens often: $A = B'B$ for some rectangular B with at least as many rows as columns. In such a case computing A and then its Choleski factor U is numerically imprudent unless performed in arithmetic at least twice as precise as the data B and the desired result Z . Otherwise U is better computed from a *QR Factorization* $B = QU$ in which Q has the same dimensions as B has and orthonormal columns; $Q'Q = I$. Similarly for V and R .