Problem 1.1
Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by:
\[
f(x, y) = (x^2 - y)^2 - 10x^2.
\]
(a) Is \( f \) convex or not?
(b) Show that \( f \) has exactly one stationary point over \( \mathbb{R}^2 \), and characterize whether it is a local minimum, a local maximum, or neither.
(c) Now consider the constrained optimization of \( f \) over the set
\[
X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}.
\]
Show that a global minimum exists, and find all points at which it is attained.

Problem 1.2
Consider the inequality-constrained problem
\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{such that } x \in C
\]
\[
C = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, \ j = 1, \ldots, m\},
\]
where \( f \) and \( g_j, j = 1, \ldots, m \) are convex and differentiable functions on \( \mathbb{R}^n \). Suppose that \( x^* \) is feasible, and the pair \((x^*, \mu^*)\) satisfy the Karush-Kuhn-Tucker necessary conditions, including the complementary slackness condition.

Using the convexity and KKT conditions, show that \([\nabla f(x^*)]^T [x - x^*] \geq 0\) for all \( x \in C \).

Problem 1.3
Let \( K \) be a non-empty, closed and convex cone, and let \( y \in \mathbb{R}^n \) be a fixed vector. In this problem we consider the projection \( \Pi_K(y) \) of \( y \) onto \( K \), which is defined by
\[
\Pi_K(y) = \arg \min_{x \in K} \|x - y\|^2,
\]
where \( \| \cdot \| \) denotes the ordinary Euclidean or \( \ell_2 \) norm.

Show that the projection \( \Pi_K(y) \in K \) satisfies the following properties:
(a) First show that \( [y - \Pi_K(y)]^T \Pi_K(y) = 0 \).
(b) Show that the difference \( \Pi_K(y) - y \) belongs to the dual cone \( K^* \).

(Hint: You can assume the result of part (a) in proving this.)
(c) Use part (b) to characterize the projections onto the following cones:

(i) The orthant cone \( K = \{ x \in \mathbb{R}^n \mid x_i \geq 0 \} \).
(ii) The semidefinite cone \( K = \{ X \in S^n \mid X \succeq 0 \} \).

**Problem 1.4**

In many applications, it is of interest to find a sparse “eigenvector” of a matrix \( \Gamma \succeq 0 \). That is, we would like to solve the optimization problem, with vector \( x \in \mathbb{R}^n \):

\[
a^* = \max x^T \Gamma x \quad \text{such that} \quad \|x\|_2 = 1, \quad \|x\|_1 \leq C.
\]

(a) Is optimization problem (1) an instance of a conic program? Why or why not?

(b) Consider the semidefinite program with matrix variable \( X \in S^n_+ \):

\[
b^* = \max \text{trace}(\Gamma X) \quad \text{such that} \quad X \succeq 0, \quad \text{trace}(X) = 1, \quad \sum_{i,j} |X_{ij}| \leq C^2.
\]

What is the relation between the two optimal values \( a^* \) and \( b^* \)?

(c) Compute the Lagrangian dual of the SDP (2). (In doing so, define the cost function \( f(X) = \text{trace}(\Gamma X) \) with \( \text{dom}(f) = \{ X \succeq 0, \text{trace}(X) = 1 \} \), so that these constraints are handled directly, and then add a Lagrange multiplier \( \mu \) only for the inequality constraint \( \sum_{i,j} |X_{ij}| \leq 1 \).)

(d) Set-up and describe the steps involved in implementing a barrier method to solve the SDP (2).

**Problem 1.5**

Consider the following questions:

(i) Show that \( f \) is convex. Calculate its domain \( \text{dom}(f) \), and the set \( \partial f(x) \) for each \( x \in \text{dom}(f) \).

(ii) Calculate the conjugate dual \( f^* \). Specify its domain \( \text{dom}(f^*) \) explicitly.

(iii) Calculate the bi-conjugate \( f^{**} \). Explain why it is (or is not) equal to \( f \).

Answer these questions for each of the following functions:

(a) \( f(x) = \exp(x) \) for \( x \in \mathbb{R} \).

(b) \( f(x) = x \log(x) - x \) for \( x > 0 \) and \( \infty \) otherwise.

(c) \( f(x) = \frac{1}{2} x^T \Gamma x \) where \( \Gamma \succeq 0 \).

(d) \( f(x) = \sup_{y \in C} y^T x \), where \( C \) is a non-empty, closed and convex set.

**Problem 1.6**

True or false: justify your answer with an argument, or an explicit counterexample if false. You may use results stated in class in order to justify your answer.
(a) The hyperbolic set $C = \{x \in \mathbb{R}^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex.

(b) Consider the optimization problem $\min_{x \in C} f(x)$, where $f$ is continuous and convex, and $C$ is a non-empty, closed and bounded polyhedron. Strong duality always holds in this case.

(c) Consider the optimization problem $\min f(x) : x \in \mathbb{R}^n, h(x) = 0$, where $f, h$ are differentiable functions on $\mathbb{R}^n$, then an optimal point must satisfy the Lagrange multiplier rule, i.e. $x^*$ is optimal only if there exists $\lambda$ such that $\nabla f(x^*) - \lambda \nabla h(x^*) = 0$

(d) Newton’s method (simplified to the problem of finding root of a differentiable function) converges to a stationary point if the starting point is sufficiently close.