Problem Set 9

Due date: Saturday, April 10 (by midnight EDT)

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be submitted on gradescope. Typesetting in LaTeX is recommended but not required. If submitting handwritten work, please make sure it is a legible and polished final draft.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

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1. (a) Suppose that there is a PCP for the language 3COLOR with the following properties:
   i. For any input graph G, the PCP proof π is supposed to be a string of N bits for $N \le |\langle G \rangle|^c$ for some fixed constant c.
   ii. The verifier picks two distinct locations of the proof, chosen randomly according to some distribution (that can depend on the input graph G), and checks that the bits in those locations are unequal.
   iii. If the input graph G is 3-colorable then there is a PCP proof $\pi(G)$ that the verifier accepts with probability 1.
   iv. If the input graph G is not 3-colorable then for *every* proof π, the verifier accepts with probability less than 1.

   Prove that this would imply that $\mathsf{P} = \mathsf{NP}$.

   (b) Now suppose we change the check made by the verifier in (a)-ii. to checking the two queried bits are equal (instead of being unequal). Which languages can have such a PCP?

2. (a) For a simple, loopless, undirected graph $G = (V, E)$, define its "square" G^2 as follows. The vertices of G^2 consist of ordered pairs of vertices of G, i.e., the vertex set is $V \times V$. Two pairs (u_1, u_2) and (v_1, v_2) are adjacent in G^2 if and only if

$$(u_1, v_1) \in E \textbf{ or } (u_2, v_2) \in E \ .$$

   Prove the following statement: For every (simple, loopless) undirected graph G, the size of the largest independent in G^2 is equal to the square of the size of the largest independent set in G.

(b) Suppose that there is a polynomial time algorithm $\mathcal{A}_{0.01}$ that on any input graph G, finds an independent set of size at least 1% of the largest independent set in G. Show how can one use $\mathcal{A}_{0.01}$ as a subroutine and design a polynomial time algorithm $\mathcal{A}_{0.99}$ that finds an independent set of size at least 99% of the largest independent set in any input graph.

<u>Hint</u>: Use the previous part.