**Carnegie Mellon University** Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2021

# Problem Set 7

Due date: Friday, March 26 (by midnight EST)

## Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be submitted on gradescope. Typesetting in LaTeX is recommended but not required. If submitting handwritten work, please make sure it is a legible and polished final draft.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

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Let $G = (V, E)$ be a (not necessarily regular) undirected graph (with no multiple edges or self loops). For a vertex $v \in V$, let $d(v)$ be its degree. Denote by $d_{\max}$ the largest degree of a vertex in $G$ (so $d_{\max} = \max_{v \in V} d(v)$), and by $d_{\text{avg}}$ the average degree of a vertex in $V$ (so $d_{\text{avg}} = \frac{1}{|V|} \sum_{v \in V} d(v)$).

Let $\lambda_1$ be the largest eigenvalue of the adjacency matrix of $G$.

(a) Prove that $d_{\text{avg}} \le \lambda_1 \le d_{\max}$.

(b) For a subset $S \subseteq V$, let $d^{(S)}_{\text{avg}}$ denote the average degree of a vertex in the subgraph of $G$ induced by $S$ (this is, the subgraph $H = (S, E')$ where $E'$ consists of all edges in $E$ both of whose endpoints lie in $S$). Prove that for all nonempty subsets $S \subseteq V$, $d^{(S)}_{\text{avg}} \le \lambda_1$.

(c) For a positive integer $k$, a graph $G = (V, E)$ is said to be $k$-colorable if there is a map $\chi : V \to \{1, 2, \ldots, k\}$ such that for all edges $(u, v) \in E$, $\chi(u) \neq \chi(v)$. That is, the vertices of $G$ can be colored with $k$ colors so that for each edge, its endpoints get two different colors. Prove that every graph $G$ is $k$-colorable for $k = \lfloor \lambda_1 \rfloor + 1$ where $\lambda_1$ is the largest eigenvalue of its adjacency matrix. (For a real number $x$, $\lfloor x \rfloor$ denotes the largest integer that is at most $x$.)

<u>Hint</u>: Given an ordering of vertices, there is a natural greedy strategy to color the vertices in that order, picking the first available color at each stage. Execute this strategy for a judiciously chosen order of vertices.