Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2021

PROBLEM SET 10
Due date: Wedesday, April 21 (by midnight EDT)

INSTRUCTIONS

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be submitted on gradescope. Typesetting in LaTeX is recommended but not required. If submitting handwritten work, please make sure it is a legible and polished final draft.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

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1. (a) Let $G = (V, E)$ be a simple undirected graph. Let d_v be the degree of a vertex $v \in V$. Prove that G has an independent set of size at least

$$\sum_{v \in V} \frac{1}{d_v + 1} \; .$$

   Hint: Consider vertices according to a random order...

   (b) Suppose G is an n-vertex simple graph that has no $(t+1)$-clique. Prove that G can have at most $\left(1 - \frac{1}{t}\right) \frac{n^2}{2}$ edges. Hint: Use part (a). The AM-HM inequality, $\sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{n^2}{\sum_{i=1}^{n} a_i}$ for positive reals a_i might be useful.

2. Let $G = (V, E)$ be an simple undirected graph on n vertices where every vertex has degree at least δ. Prove that there exists a subset $D \subseteq V$ of size at most

$$\left(\frac{1 + \log(\delta + 1)}{\delta + 1} \right) \cdot n$$

   such that every vertex in $V \setminus D$ is adjacent to some vertex in D.

   Suggestion: Pick a random subset R by including each vertex in R with some suitable probability p (which you will optimize). Take D to be R plus those vertices with no neighbor in R. The inequality $1 + x \leq e^x$ for all $x \in \mathbb{R}$ might be handy in your computations.