Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 11

Due date: Sunday, April 12

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be typeset in LaTeX and emailed to the TA.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

1. In this exercise, we will prove a special case of the fact that good spectral expansion implies good vertex expansion. Let $G = (V, E)$ be a d-regular graph with $d > 8$. Let A be its adjacency matrix and $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Suppose that G is a Ramanujan graph i.e. $\lambda(G) = \max\{|\lambda_2|, |\lambda_n|\} = 2\sqrt{d-1}$.

   Let $S \subseteq V$ be an arbitrary set of vertices such that $|S| = \alpha n$ with $\alpha = \frac{8}{d}$. Let $N(S)$ be the set of the neighbors of S i.e. $N(S) = \{v : \exists u \in S, (u, v) \in E\}$. Prove that $|N(S) \setminus S| \geq \frac{n}{2} - |S|$.

   Hint: Use the expander mixing lemma.

2. Let $G = (V, E)$ be a d-regular graph. Let $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A of the graph G. Let $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$. We have discussed in class that the lower λ is, the better expander G is. However, λ cannot be arbitrarily small. Prove that $\lambda \geq \sqrt{\frac{d(n-d)}{n-1}}$.

   If d is a constant, for large enough n, this proves that $\lambda \gtrsim \sqrt{d}$. This is not far from the tight bound of $2\sqrt{d-1}$.

   Hint: Use the fact that the sum of diagonal elements of a matrix (also known as the trace of a matrix) is equal to the sum of its eigenvalues.