Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 10

Due date: Saturday, April 4

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be typeset in LaTeX and emailed to the TA.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

1. (a) Prove that every n-vertex graph that does not have a triangle (3-clique) has an independent set of size at least $\lfloor\sqrt{n}\rfloor$.

   Hint: Consider two cases, based on the maximum degree of a vertex in the graph.

   (b) Let $\mathbb{G}(n,p)$ denote the distribution on n-vertex graphs where each edge is included independently with probability p. (We considered $\mathbb{G}(n,1/2)$ in the lecture.)

   i. Consider a random graph G sampled from $\mathbb{G}(n,p)$. What is the expected number of triangles in G?

   ii. Now suppose $p = n^{-2/3}$. Prove that a random graph G sampled from $\mathbb{G}(n,p)$ has no independent set of size $n^{2/3}\log n$ with probability approaching 1 for large n. (The inequality $1-p \le e^{-p}$ for $p \in (0,1)$ might be handy.)

   iii. Using the above two parts, prove that for large enough n, there exist an n-vertex graph that is triangle-free *and* does not have an independent set of size $cn^{2/3}\log n$, for some absolute constant c (that is independent of n).

   (FYI, it is known that there exist triangle-free graphs with no independent sets of size $\Theta(\sqrt{n\log n})$, and that this is best possible up to constant factors, but these optimal bounds are much harder to prove.)