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Topic: The Influence of Boolean Variables
Date: 3/19/99
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Motivation

Suppose we have a function $f : \{0,1\}^n \to \{0,1\}$. We want to understand the influence of variables. In terms of the boolean cube, the question is: pick a variable i. What fraction of the edges are bichromatic? Or, in other words, for what fraction of the inputs can changing i change f?

This is based on the proof presented by Jeff Kahn, Gil Kalai, and Nathan Linial in a paper entitled "The Influence of Variables on Boolean Functions", published in IEEE 1988.

Define β_i = fraction or bichromatic edges in dimension i

If $S \subseteq 1, \ldots, n$, The influence of S denoted β_S = fraction of S-cubes that are bichromatic.

This lecture will show that there exists some β_i with influence at least $\frac{\log n}{n}$.

Here's some examples: (ρ is the fraction of 1's.)

function	ρ	max β_i	$1/n \sum_i \beta_i$
AND	2^{-n}	2^{1-n}	2^{1-n}
MAJ	$1/2$	$\Theta(1/\sqrt{n})$	$\Theta(1/\sqrt{n})$
parity	$1/2$	1	1
x_1	$1/2$	1	$1/n$
from [BL]	$1/2$	$\Theta(\log n/n)$	$\Theta(\log n/n)$

Example from BL

This example is from Ben-Or and Linial. It is a function where each variable has influence $\Theta(logn/n)$. The function is an and of ors, in other words, $f = y_1 \wedge y_2 \wedge \cdots \wedge y_m$ where m is about $\log n/n$. The y_i's are ors of about $\log n - \log\log n$ variables. This function is balanced. If the input is chosen randmoly, the probablility that $f = 1$ is about $1/2$. The maximum influsence is $\Theta(logn/n)$, and the average is the same.

Our goal is to show that for any f, there exists a small $(\log n/n)$ sets with influence aproaching 1. To do this, we will show that single variable is substantial.

Theorem 1 *For any f there exists i such that $\beta_i = \Theta\left(\frac{\rho \log n}{n}\right)$.*

We can't just argue simple from the average, since a variable could have enormous influence while the average was very small. (For example, $f = x_1$ above). So, we'll use something average-like that increases when the set is unbalanced.

Create a n functions, each indicating the amount of influence in dimension i. $f^i : \{0,1\}^n \to \{-1,0,1\}$

$f^i(s) = f(s) - f(s \oplus i)$ Where $\oplus$ to be the symmetric difference if we think of s as a set, or exclusive or if we think of s as a bit vector.

We'll be using Fourier transforms. Define $\mu_S(R) = 2^{-n/2}(-1)^{|S \cap R|}$ if we think of R and S as sets, or as $2^{-n/2}(-1)^{S \cdot R}$ if we think of them as bit vectors. Denote the Fourier coefficient of f as α_S.

Note that under the above definition of μ_S, $\|f\|_2^2 = \sum_S \alpha_S^2$ (by Parseval's identity). Furthermore, since $\|f\|_2^2$ essentially counts the number of times f is one, $\rho = 2^{-n}\|f\|_2^2$.

$$\begin{aligned}
\alpha_S^i &= < \mu_S, f^i >= 2^{-n/2} \sum_R f(R)(-1)^{|S \cap R|} \\
&= 2^{-n/2} \sum_R (f(R) - f(R \oplus i))(-1)^{|S \cap R|} \\
&= 2^{-n/2} \sum_R f(R) \left((-1)^{|S \cap R|} - (-1)^{|S \cap (R \oplus i)|} \right)
\end{aligned}$$

$= 0$ if $i \notin S$, and $2\alpha_S$ otherwise.

Since $\|f^i\|_2^2$ counts the bichromatic edges in dimension i (to see this, notice that $f^i(S)$ is non-zero iff and only if changing i changes the value of f), $2^n \beta_i = \|f^i\|_2^2$, or $\beta_i = 4 \sum_{S : i \in S} \alpha_S^2$.

We can get the arithmetic average from $\sum_i \beta_i = 4 \sum_S |S| \alpha_S^2$. If we could somehow bound the contribution from the small sets (sets of size less than $\log n$), then we'd be done. As an interesting side note, $\sum \alpha_s^2 = 2^n \rho$ implies isoparametric inequality for the cube.

Bounding the Contribution from Small Sets

$$T : \mathbf{R}^2 \to \mathbf{R}^2$$

$\begin{bmatrix} \frac{1+\epsilon}{2} & \frac{1-\epsilon}{2} \\ \frac{1-\epsilon}{2} & \frac{1+\epsilon}{2} \end{bmatrix}$ T has eigan vectors $u_o = \frac{1}{\sqrt{2}}(1,1)$, eigan value 1, and $u_1 = \frac{1}{\sqrt{2}}(1,-1)$, value ϵ. T is therefore a contraction map.

Now, define:

$$\mathcal{M}_r(v) = \left(1/k \sum |v_i|^r \right)^{1/r}$$

where $v = (v_1, v_2, \ldots, v_k)$

We state the following without proof:

Beckner 1 *T is a non-expanding map under this operator. In other words, "T is of norm 1 as a map $L_{1+\epsilon^2} \to L_2$"*

The point of this is that $\mathcal{M}_{1+\epsilon^2} \geq \mathcal{M}_2(Tv)$, for $0 < \epsilon < 1$.

In other words, $\mathcal{M}_{1+\epsilon^2}(v) \geq \mathcal{M}_2(Tv)$ for $0 < \epsilon < 1$, T.

Define

$$T^n = \underbrace{T \otimes T \otimes \cdots \otimes T}_{n}$$

which operates on

$$\underbrace{\mathbf{R}^2 \otimes \mathbf{R}^2 \otimes \cdots \otimes \mathbf{R}^2}_{n}$$

Now, T^n has eigenvectors μ_S and $\lambda_S = \epsilon^{|S|} 1^{n-|S|} = \epsilon^{|S|}$.

We state the following without proof.

Beckner 2 *If T is norm 1 $L_{1+\epsilon^2} \to L_w$, then so is T^n.*

Note that T^n operates on basically $\mathbf{R}^{2^n}$. You can think of a function that takes $\{0,1\}^n$ to $\mathbf{R}$ as a vector where the first element of the vector is $f(0,0,0,\ldots,0)$, the second is $f(0,\ldots,0,1)$ and so on. In that sense, T^n acts on functions from $\{0,1\}^n$. This means that $\mathcal{M}_{1+\epsilon^2}(f^i) \geq \mathcal{M}(T^n f^i)$.

Now we are ready to do some calculations.

$$\begin{aligned}
\mathcal{M}_{1+\epsilon^2}(f^i) &= \underbrace{\left(2^{-n}\sum_s f^i(S))^{1+\epsilon^2}\right)^{\frac{1}{1+\epsilon^2}}}_{\beta_i^{\frac{1}{1+\epsilon^2}}} \\
&\geq \mathcal{M}_2(T^n f^i) \\
&= (2^{-n}\sum_s (T^n f^i)^2)^{1/2}
\end{aligned}$$

Recall that $f^i(S)$ is either 0, 1 or -1. This means that $|f^i(S)|^{1+\epsilon^2}$ will be either 0 or 1. That's why we can say, as we indicated by the underbrace, that $\left(2^{-n}\sum_s f^i(S))^{1+\epsilon^2}\right)^{\frac{1}{1+\epsilon^2}} = \beta_i^{\frac{1}{1+\epsilon^2}}$. Getting back into the above equations, we have

$$\begin{aligned}
\beta_i^{\frac{1}{1+\epsilon^2}} &\geq 2^{-n/2}\|T^n f^i\|_2 \\
&= 2^{-n/2}\sum_{S:i\in S}(\epsilon^{|S|}4\alpha_s)^2)^{1/2} \\
&= 2^{1-n/2}(\sum_{S:i\in S}(\epsilon^{|S|}\alpha_s)^2)^{1/2}
\end{aligned}$$

Now, square and sum

$$\sum_i \beta_i^{\frac{2}{1+\epsilon^2}} \geq 2^{2-n}\sum_S \epsilon^{2|S|}\alpha_S^2$$

$$\geq \quad 2^{2-n}\epsilon^{2b} \sum_{S:|S|\leq b} |S|\alpha_S^2$$

$$\geq \quad 2^{2-n}\epsilon^{2b} \left(\sum_{S:|S|\leq b} \alpha_s^2 - \alpha_\emptyset^2 \right) \tag{1}$$

If you'll remember back a few pages, we had $\sum_i \beta_i = 4\sum_S |S|\alpha_S^2$. This implies

$$\sum_i \beta_i \geq 4 \sum_{S>b} b\alpha_S^2 \geq 4b \sum_{S>b} \alpha_S^2 \tag{2}$$

Clearly,

$$\sum_{S:S\leq b} \alpha_S^2 + \sum_{S:S>b} \alpha_S^2 = \sum_S \alpha_S^2 . \tag{3}$$

Substituting equation 1 and equation 2 into the equation 3, we get:

$$2^n 1/b \sum_{S:|S|>b} \beta_i + 2^n \epsilon^{-2b} \sum_{S:|S|\leq b} \beta_i^{\frac{2}{\epsilon^2}} + \alpha_\emptyset \geq \sum_S (\alpha_S^2)$$

.

Moving $\alpha_\emptyset$ to the other side we get and dividing by 2^n:

$$1/b \sum_{S:|S|>b} \beta_i + \epsilon^{-2b} \sum_{S:|S|\leq b} \beta_i^{\frac{2}{1+\epsilon^2}} \geq 2^{-n} \sum_S (\alpha_S^2) - 2^{-n}\alpha_\emptyset^2 \tag{4}$$

Recall that $\sum_S \alpha_S^2 = \|f\|$ by Parseval's identity, and so is equal to $2^n\rho$. Also, notice that $\alpha_\emptyset = 2^{-n/2}\sum f$. This means that $\alpha_\emptyset = 2^{n/2}\rho$. Hence, $\alpha_\emptyset^2 = 2^n\rho^2$, or $2^{-n}\alpha_\emptyset^2 = \rho^2$. So, the right side of the equation 4 becomes $\rho - \rho^2$. We can assume, without loss of generality, that $\rho \geq 1/2$. (The influence of a variable is the same in both f and f complement.) An easy calculation shows that when $\rho \leq 1/2$, $\rho - \rho^2 \geq \rho/2$. Apply the above to equation 4:

$$1/b \sum_{S:|S|>b} \beta_i + \epsilon^{-2b} \sum_{S:|S|\leq b} \beta_i^{\frac{2}{1+\epsilon^2}} \geq \rho/2$$

Now, choose $\epsilon^2 = 1/2$, $b = c\log n$ and $c = \frac{1}{10\log 2}$.

Then, we know that either $1/b\sum \beta_i \geq p/4$, or $\epsilon^{-2b}\sum |\beta_i|^{\frac{2}{1+\epsilon^2}} \geq \rho/4$. Suppose the first is true. Then $\sum \beta_i \geq \rho b/4$, so $\sum \beta_i \geq c\rho \log n/4$. If this is true, then for some i, $\beta_i \geq \frac{cp\log n}{4n}$, and we are done.

The other case is slightly more complicated. But we know:

$$\left(1/n \sum \beta_i^{\frac{2}{1+\epsilon^2}}\right)^{\frac{1+\epsilon^2}{2}} \geq (n^{-1}\epsilon^{2b}p/4)^{\frac{1+\epsilon^2}{2}} \geq \left(p/4n^{-1-2c\log 1/\epsilon}\right)^{\frac{1+\epsilon^2}{2}}$$

We want $(1+2c\log 1/\epsilon)^{\frac{1+\epsilon^2}{2}} < 1$. This is true for ϵ and c as chosen. Call this value $1-\delta$. Then we can apply $\mathcal{M}$ to the vector of β_i's, $\mathcal{M}_{\frac{2}{1+\epsilon^2}}\beta \geq n^{\delta-1}$, and so, since $\mathcal{M}_1 \geq \mathcal{M}_{\frac{2}{1+\epsilon^2}}$ there exists i with $\beta_i \geq n^{\delta-1}$, and $n^{\delta-1}$ is asymptotically bigger than $\log n/n$, so we are done.