1 Larmor Precession

Turning on a magnetic field $\vec{B}$, the qubit state rotates. There are two steps to understanding this process, essentially the same steps we make to understand any quantum process:

1. Find $\hat{H}$
2. Solve Schrödinger equation

For the second step, we first solve the "time-independent" Schrödinger equation; that is, we find energy eigenstates

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle \ .$$

The "time-dependent" Schrödinger equation

$$i\hbar\tfrac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

has solution

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle \ .$$

Expanding $|\psi(t=0)\rangle = \sum_n c_n|\psi_n\rangle$, we get

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|\psi_n\rangle \ .$$

(This assumes that $\hat{H}$ is time-independent. If the Hamiltonian is itself a function of t, $\hat{H} = \hat{H}(t)$, then we must directly solve the time-dependent Schrödinger equation.)

1.1 Find $\mathscr{H}$

Assume there is only potential energy, not kinetic energy. Classically, $E = -\vec{\mu}\cdot\vec{B}$. Quantumly, the magnetic moment is in fact a vector *operator*, $\hat{\vec{\mu}} = \frac{gq}{2m}\hat{\vec{S}} = -\frac{e}{m}\hat{\vec{S}}$. Hence we set the quantum Hamiltonian to be

$$\hat{H} = \tfrac{e}{m}\hat{\vec{S}}\cdot\vec{B} \ .$$

We may choose our coordinate system so $\vec{B} = B\hat{z}$; then

$$\hat{H} = \tfrac{eB}{m}\hat{S}_z \ .$$

1.2 Solve Schrödinger Equation

Following the recipe we gave above, we start by finding the eigendecomposition of $\hat{H}$. The eigenstates of $\hat{H}$ are just those of $\hat{S}_z$: $|0\rangle$ (up) and $|1\rangle$ (down). The corresponding eigenenergies are $E_0 = \frac{eB}{2m}\hbar$, $E_1 = -\frac{eB}{2m}\hbar$.

Next we solve the time-dependent Schrödinger equation. Write

$$|\psi(t=0)\rangle = \alpha|0\rangle + \beta|1\rangle \quad .$$

Then

$$\begin{aligned} |\psi(t)\rangle &= \alpha e^{-i\frac{eB}{2m}t}|0\rangle + \beta e^{i\frac{eB}{2m}t}|1\rangle \\ &\propto \alpha|0\rangle + \beta e^{i\frac{eB}{m}t}|1\rangle \quad , \end{aligned}$$

where the proportionality is up to a global phase. On the Bloch sphere,

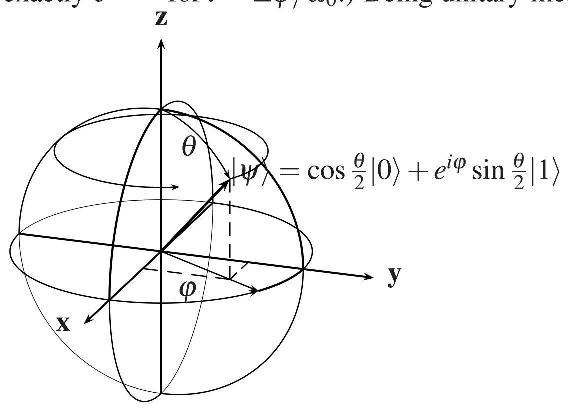
$$|\psi(t=0)\rangle = \cos\tfrac{\theta}{2}|0\rangle + \sin\tfrac{\theta}{2}e^{i\varphi}|1\rangle$$

evolves to

$$|\psi(t)\rangle = \cos\tfrac{\theta}{2}|0\rangle + \sin\tfrac{\theta}{2}e^{i(\varphi+\frac{eB}{m}t)}|1\rangle \quad .$$

Thus the state rotates counterclockwise around the z axis, at frequency $\omega_0 \equiv \frac{eB}{m}$ (ω_0 is known as the cyclotron frequency, since it is the same frequency with which a classical e^- cycles in a magnetic field, due to the Lorentz force).

Therefore $\hat{R}_z(\Delta\varphi) = e^{-i\frac{\hat{S}_z}{\hbar}\Delta\varphi}$ is a unitary operation which rotates by $\Delta\varphi$ about the z axis. (Proof: $\hat{R}_z(\Delta\varphi)$ is exactly $e^{-i\frac{\hat{H}}{\hbar}t}$ for $t = \Delta\varphi/\omega_0$.) Being unitary means $\hat{R}_z(\Delta\varphi)^\dagger = \hat{R}_z(\Delta\varphi)^{-1} = \hat{R}_z(-\Delta\varphi)$.



Aligning $\vec{B}$ with the z axis rotates the spin about the z axis. Each state is restricted to the line of latitude it starts on, as illustrated above. For a more general rotation about a different axis, simply point the $\vec{B}$ field in a different direction. For example, the unitary operator

$$\hat{R}_n(\Delta\gamma) = e^{-i\frac{\hat{\vec{S}}\cdot\hat{n}}{\hbar}\Delta\gamma}$$

rotates by $\Delta\gamma$ about the axis $\hat{n}$. To achieve this unitary transformation, set $\vec{B} = B\hat{n}$ for exactly time $t = \Delta\gamma/\omega_0$.

Any unitary transformation on a single qubit, up to a global phase, is a rotation on the Bloch sphere about some axis; mathematically, this is the well-known isomorphism $SU(2)/\pm 1 \cong SO(3)$ between 2×2 unitary matrices up to phase and 3×3 real rotation matrices. Hence Larmor precession, or spin rotation, allows us to achieve any single qubit unitary gate. While theoretically simple, Larmor precession can unfortunately be inconvenient in real life, mostly because of the high frequencies involved and the susceptibility to noise. A more practical method for achieving rotations on the Bloch sphere is spin resonance, which we will describe next.

2 Spin Resonance

Spin resonance comes in many varieties, for example ESR (for electrons), μSR (for muons), and NMR/MRI (nuclear magnetic resonance/magnetic resonance imaging).

Assume we start with a number of randomly oriented spins.

In spin resonance, we start by turning on a large magnetic field $\vec{B} = B_0\hat{z}$, on the order of kT, in order to align spins against this large field.

In a magnetic field, electrons with spins aligned against the field have an energy advantage of $\Delta E = \frac{e}{m}B\hbar$ over spins aligned with the field. Therefore, the spins will slowly reach a thermal (statistical) equilibrium by aligning against the field to minimize energy. (This thermal process occurs on a slower time scale than the Larmor precession in a $\vec{B}$ field.)

The trick is to then turn on small, oscillating magnetic field $\vec{B} = B_1 \cos\omega_0 t\hat{x}$. Let's analyze what happens:

First we find $\hat{H}$:

$$\begin{aligned} \hat{H}(t) &= \frac{e}{m}\vec{B}\cdot\hat{\vec{S}} \\ &= \frac{e}{m}(B_0\hat{S}_z + B_1\cos\omega_0 t\hat{S}_x) \\ &= \frac{\hbar e}{2m}\begin{pmatrix} B_0 & B_1\cos\omega_0 t \\ B_1\cos\omega_0 t & -B_0 \end{pmatrix} \quad , \end{aligned}$$

where we have substituted the matrix representations for $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Since $\hat{H}$ depends on time, we directly substitute it into the *time-dependent* Schrödinger equation $i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$. For $|\psi(t)\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$, we get the differential equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \frac{e\hbar}{2m}\begin{pmatrix} B_0 & B_1\cos\omega_0 t \\ B_1\cos\omega_0 t & -B_0 \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \quad .$$

More next time!