1. Consider two quantum systems each with two qubits, where the first is in the state $1/2\big|00\big\rangle + i/2\big|10\big\rangle + 1/\sqrt{2}\big|11\big\rangle$, and the second is in the state $1/\sqrt{3}\big|00\big\rangle + i/\sqrt{3}\big|10\big\rangle - 1/\sqrt{3}\big|01\big\rangle$. What is the state of the composite system? If we apply a CNOT to the first system and the transformation $H \otimes Z$ to the second, what is the new state of the system. Write out the matrix corresponding to the unitary transformation you applied to the second system.

2. Suppose that Alice has two qubits in an entangled state $|\psi\rangle \in \mathscr{C}^2 \otimes \mathscr{C}^2$. If she teleports each of her two qubits to Bob using the teleportation protocol presented in lecture, can Bob faithfully reconstruct the (entangled) state of Alice's two qubits? Justify your answer.

3. The uncertainty principle bounds how well a quantum state can be localized simultaneously in the standard basis and the Fourier basis. In this question, we will derive an uncertainty principle for a discrete system of n-qubit.

   Let $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$ be the state of an n-qubit system. A measure of the spread of $|\psi\rangle$ is $S(|\psi\rangle) \equiv \sum_x |\alpha_x|$. For example, for a completely localized state $|\psi\rangle = |y\rangle$ ($y \in \{0,1\}^n$), the spread is $S(|\psi\rangle) = 1$. For a maximally spread state $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle$, $S(|\psi\rangle) = 2^n \cdot \frac{1}{\sqrt{2^n}} = \sqrt{2^n}$.

   a) Prove that for any quantum state $|\psi\rangle$ on n qubits, $S(|\psi\rangle) \leq 2^{n/2}$.

   b) Suppose that $|\alpha_x| \leq a$ for all x. Prove that $S(|\psi\rangle) \geq \frac{1}{a}$.

   Now let $H^{\otimes n}|\psi\rangle = \sum_x \beta_x |x\rangle$, where (by homework 2) $\beta_x = \frac{1}{2^{n/2}} \sum_y (-1)^{x\cdot y} \alpha_y$. ($x \cdot y \equiv \sum_{i=1}^n x_i y_i$.)

   c) Prove that for all y, $|\beta_y| \leq \frac{1}{2^{n/2}} S(|\psi\rangle)$.

   d) Prove the uncertainty relation $S(|\psi\rangle) S(H^{\otimes n}|\psi\rangle) \geq 2^{n/2}$. Justify why it makes sense to call this an uncertainty relation.

4. Show that the trace of an operator is independent of the basis in which it is evaluated.