Quantum computing Set 1 Solutions

Problem 1

(1)

$$E = (+1)\, \Pr(\text{measuring } |0\rangle) + (-1)\, \Pr(\text{measuring } |1\rangle)$$

$$= |\langle 0|\psi\rangle|^2 - |\langle 1|\psi\rangle|^2$$

$$= \langle\psi|0\rangle\langle 0|\psi\rangle - \langle\psi|1\rangle\langle 1|\psi\rangle$$

$$= \langle\psi|\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)|\psi\rangle$$

$$= \langle\psi|Z|\psi\rangle.$$

(2) Since $X = |+\rangle\langle +| - |-\rangle\langle -|$, by analogy, measure in $|+\rangle, |-\rangle$ basis and associate $|+\rangle$ with $+1$ and $|-\rangle$ with -1.

Problem 2

In the first interpretation of CHSH, on input $x, y \in \{0,1\}$ our score was $E(u_x \oplus v_y \oplus xy)$ $u_x, v_y \in \{0,1\}$. To switch interpretation, we can scale our score to $E(2(u_x \oplus v_y + xy) - 1)$ changing the game. Notice $2(a) - 1 = -(-1)^a$ for $a \in \{0,1\}$.

So its equivalent to our score being

$$E\left(-(-1)^{u_x \oplus v_y \oplus xy}\right).$$

Let's redefine $u_x, v_y \in \{\pm 1\}$ by $u_x = (-1)^{u_x}$, $v_y = (-1)^{v_y}$.

Expanding E over $\Pr(x,y) = \frac{1}{4}$ and scaling by 4 our score is equivalent to

$$\sum_{xy} E\left(-(-1)^{xy} u_x v_y\right) = E(u_0 v_0) + E(u_1 v_0) + E(u_0 v_1) - E(u_1 v_1).$$

Problem 2 (cont.)

Our transformation of the score was $4(2(\cdot)-1) = 8(\cdot)-4$.

So the quantum score on the original CHSH is now

$$8\cos^2 \pi/8 - 4 = 8\cos^2 \pi/8 - 4\cos^2 \pi/8 - 4\sin^2 \pi/8$$

$$= 4\left(\cos^2 \pi/8 - \sin^2 \pi/8\right) = 2\sqrt{2}.$$

In any det. classical strategy, either $v_0 + v_1 = 0$ or $v_0 - v_1 = 0$. Therefore

$$u_0(v_0+v_1) + u_1(v_0 - v_1) \leq 2.$$

A randomized strategy is a linear combination of det. strategies.

Problem 3

① Let $a_x, b_x, c_x, a_y, b_y, c_y$ be the deterministic answers by Alice, Bob and Charlie for inputs x, y. Then in order to satisfy w.p. 1,

$$a_x b_x c_x = 1$$
$$a_y b_y c_x = -1$$
$$a_y b_x c_y = -1$$
$$a_x b_y c_y = -1$$

Multiplying all 4 equations together

$$\underbrace{a_x^2 a_y^2 b_x^2 b_y^2 c_x^2 c_y^2}_{\geq 0} = -1 \quad \leftarrow \text{impossible.}$$

For 3/4 strategy, everyone always answers -1.

Problem 3 (cont.)

② wlog assume $a_x, b_x, c_x, a_y, b_y, c_y$ are now fns $\Omega \to \{\pm 1\}$ for some set Ω and assume they share a random sample $\sigma \in \Omega$. To win with pr 1, $\forall\, \sigma \in \Omega$ with $\mathrm{pr}(\sigma) > 0$, $a_x(\sigma), b_x(\sigma), \ldots, c_y(\sigma)$ must satisfy the equations on the previous page which is impossible. Therefore, randomness does not yield pr 1 success.

③ Since the eigenvalues of a tensor product are the products of the eigenvalues of the terms,

$$\langle X \otimes X \otimes X \rangle_\gamma = \Pr(\text{win}) - \Pr(\text{lose}) = 2\Pr(\text{win}) - 1$$

since we win if the product of the answers is 1. Suffices then to verify $\langle X \otimes X \otimes X \rangle_\gamma = 1$.

④ Analagously, check that $\langle X \otimes Y \otimes Y \rangle_\gamma = -1$.

⑤ Let $\Pi_\sigma : |a_1, a_2, a_3\rangle \mapsto |a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\rangle$. Easy to check $\Pi_\sigma |\gamma\rangle = |\gamma\rangle$.

Notice that $Y \otimes X \otimes Y = \Pi_{(12)} X \otimes Y \otimes Y\, \Pi_{(12)}$. Therefore,

$$\langle Y \otimes X \otimes Y \rangle_\gamma = \langle \gamma | \Pi_{(12)} X \otimes Y \otimes Y\, \Pi_{(12)} | \gamma \rangle = \langle \gamma | X \otimes Y \otimes Y | \gamma \rangle = -1.$$

Problem 3 (cont.)

Therefore, wins with pr 1. Similar argument holds for $Y \otimes Y \otimes X$.

Problem 4

① This actually holds for general d-dim maximally entangled state: $\frac{1}{\sqrt{d}} \sum_i |i\rangle|i\rangle$.

$$\mathcal{U} \otimes \mathbb{I} |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_i \mathcal{U}|i\rangle \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i,j} \underbrace{|j\rangle\langle j|}_{\mathbb{I}} \mathcal{U}|i\rangle \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i,j} |j\rangle \otimes |i\rangle \langle j|\mathcal{U}|i\rangle$$

b.c. $(\langle j|\mathcal{U}|i\rangle)^T = |i\rangle^T \mathcal{U}^T \langle j|^T$

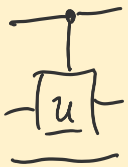
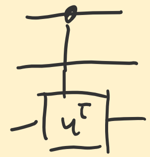
$= \langle i|\mathcal{U}^T|j\rangle$

$$= \frac{1}{\sqrt{d}} \sum_{i,j} |j\rangle \otimes \underbrace{|i\rangle \langle i|}_{\mathbb{I}} \mathcal{U}^T|j\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_j |j\rangle \mathcal{U}^T |j\rangle$$

$$= \mathbb{I} \otimes \mathcal{U}^T |\Phi\rangle.$$

Problem 4 (cont.)

Let $\mathrm{Ctrl}_{12}\text{-}U =$  and $\mathrm{Ctrl}_{13}\text{-}U^T =$ . $\Longrightarrow$

$$\mathrm{Ctrl}_{12}\text{-}U\,|0\rangle|\Phi\rangle = |0\rangle\, \mathbb{I}_2 |\Phi\rangle$$
$$= |0\rangle|\Phi\rangle$$
$$= \mathrm{Ctrl}_{13}\text{-}U^T\,|0\rangle|\Phi\rangle$$

$$\mathrm{Ctrl}_{12}\text{-}U\,|1\rangle|\Phi\rangle = |1\rangle\, U_2 |\Phi\rangle$$
$$= |1\rangle\, U_3^T |\Phi\rangle$$
$$= \mathrm{Ctrl}_{13}\text{-}U^T\,|1\rangle|\Phi\rangle$$

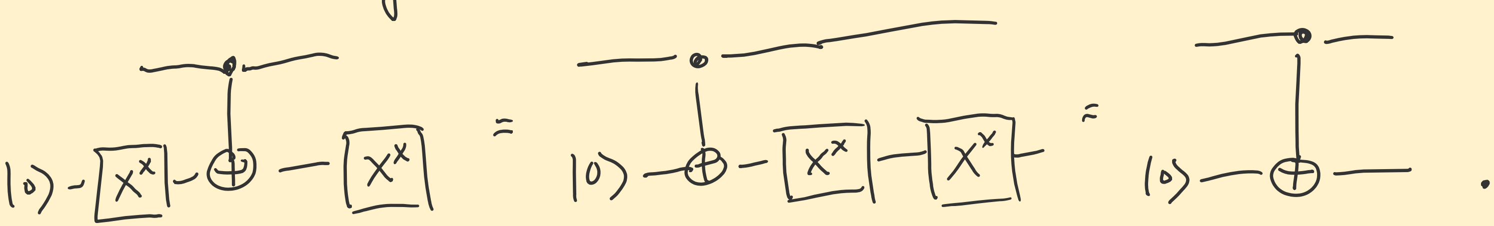
Holds by linearity for general state on 1^{st} qubit.

Problem 4 (cont.)

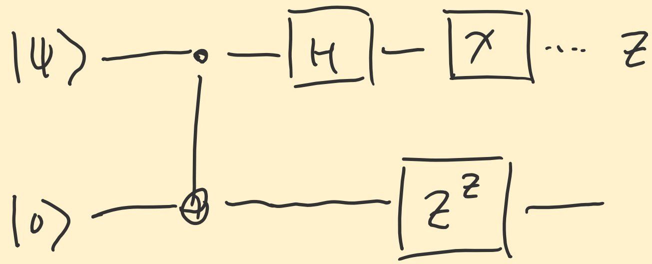
② We apply (part 1) to the EPR state to move the CNOT to the 3^{rd} qubit.

Now we are measuring one qubit of the EPR state in the computational basis. The principle of deferred measurement cuts both ways so the circuit is equivalent to with pr $1/2$ running the ckt in eq.(3) for each $x \in \{0,1\}$.

③ $|x\rangle -$ $=$ $|0\rangle - \boxed{X^x} -$. As CNOT = Ctrl$-X$, its easy to see that commutation gives us



Problem 4 (cont.)

④ 

is equivalent to $|\psi\rangle|0\rangle =$

$$(\alpha|0\rangle + \beta|1\rangle)\otimes|0\rangle \to \alpha|00\rangle + \beta|11\rangle$$

$$\to \frac{1}{\sqrt{2}}\left(\alpha|00\rangle + \alpha|10\rangle + \beta|01\rangle - \beta|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle Z|\psi\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \sum_z |z\rangle Z^z|\psi\rangle$$

measuring z collapses the state to $Z^z|\psi\rangle$ so applying Z^z produces $|\psi\rangle$.