Lecture Notes 10/07/2016

SCRIBES: JOSEPH BROZ, ERIK URBAN

Prof: Umesh Vazirani

October 13, 2016

1 QMA COMPLETENESS OF LOCAL HAMILTONIANS

Consider a system of m qubits in a line $[(\mathbb{C}^2)^{\otimes m}]$

$$\underbrace{\bullet \;\; \bullet \;\; \bullet \ldots \bullet}_{m}$$

subject to a Hamiltonian H which can be broken down into several "k-local" hamiltonian terms (not, generally, mutually compatible):

$$H = \sum_{j=1}^{n} H_j \tag{1.1}$$

where the H_j's are local in the sense that they act as the identity on all but k of the qubits, so that we can write $H_j = E_j \otimes I$. Additionally, assume the scaling $0 \le ||E_j|| \le 1$.

Given H, a typical question asked is: What's the eigenenergy λ of its ground state $|\Omega\rangle$?

The k-local Hamiltonian problem (k-LH) is defined as follows: Given H and $a < b \in [0,1]$ s.t. $b - a \ge \text{poly}(m)$, is $\lambda \le a$ or is $\lambda \ge b$, promised that one of these conditions holds.

1.1 CORRESPONDENCE TO MAX-3SAT

MAX-3SAT on m variables is defined as:
Given $f(x_1, ...x_m) = \wedge_{i=1}^{n}(x_{i1} \vee x_{i2} \vee x_{i3})$, where each x_{ij} is either a variable or its negation, find an assignment that satisfies the most disjunctive clauses of f.

Now, for the local Hamiltonian problem suppose that $k = 3$ and that the E_j's can all be written as:

$$E_j = |u\rangle\langle u| = \begin{bmatrix} 0 & & & \\ & \ddots & & \mathbf{0} \\ & & 1 & \\ & \mathbf{0} & & \ddots \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{8x8}, \tag{1.2}$$

for some $|u\rangle$ in the 3-qubit computational basis. The eigenvectors of H will also be classical states.

Take each E_j to represent a disjunctive clause with an eigenvalue of 1 corresponding to false and an eigenvalue of 0 corresponding to true. Then consider a classical state $|x\rangle = (\mathbb{C}^2)^{\otimes m}$. The energy of $|x\rangle$ is the number of violated clauses E_j. From this we can see that MAX-3SAT is reducible to 3-LH, which is known to be NP-complete.

It's reasonable to suspect that 3-LH is harder than MAX-3SAT since this case assumes all E_j's project onto the same basis (H_j's commute) but that is not necessarily true for the general 3-local Hamiltonian problem.

1.2 Arthur-Merlin Protocol

Consider a game between two players, Merlin (M) who is all powerful, and Arthur (A) who is polynomial time bound. They both have access to a Boolean-valued function $f(x_1, ...x_m)$ subject to n constraints. M generates an assignment $(a_1, ..., a_m)$ and sends it to A. A's job is to determine whether or not the assignment satisfies f with high probability. If A can do this using a probabilistic polynomial time computation, then we say that $f \in$ MA. (This differs from NP in A's ability to use randomness).

QMA is the quantum analogue of MA. The idea is the same, but now A has access to a probabilistic polynomial time quantum verifier.

1.3 Is the local hamiltonian setup QMA?

WLOG assume the E_j's are projections matrices. Restating LH; Given:

$$H_j, a, b : |b - a| \geq 1/\text{poly}(m) \tag{1.3}$$

we'd like to decide whether $\lambda \leq a$ or $\lambda \geq b$ where $|b - a|$ is determined by our desired precision.

Now, M sends N copies of $|\Omega\rangle$ to A, who uses these to estimate $\lambda = \sum_{i=1}^{n} \langle\Omega|H_j|\Omega\rangle$. A picks j at random and measures H_j.

$$\langle H_j\rangle = \langle\Omega|H_j|\Omega\rangle \tag{1.4}$$

Since the E_j's are projections, the outcome of this measurement is a coin flip with bias λ/n. We want a precision of $\epsilon = |b-a|/n$, which requires $1/\epsilon^2$ flips. Therefore, the number of required operations is

$$N = \mathcal{O}\left(\frac{n^2}{(b-a)^2}\right) \tag{1.5}$$

1.4 Is LH QMA complete?

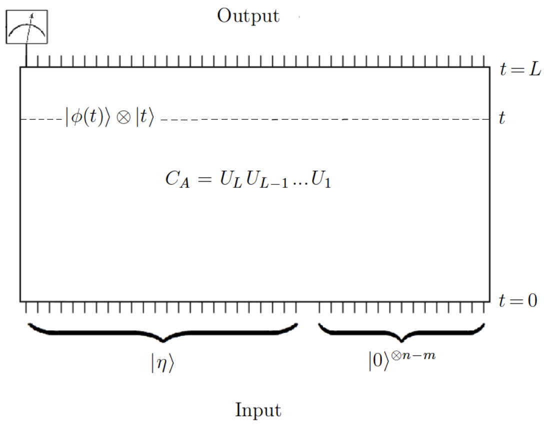

Figure 1.1:

If the universal QMA problem reduces to LH, then LH is QMA-complete. I.e. given a quantum circuit $C_A = U_L U_{L-1} \dots U_1$ and the promise that one of the following two conditions holds

1. $\exists|\eta\rangle$ s.t. C_A accepts with probability greater than or equal to $1-\epsilon$
2. $\forall|\eta\rangle$, C_A accepts with probability less than or equal to ϵ,

To show this, we need to design a sum of local Hamiltonians H such that for some a, b:

1. $\lambda \leq a$ iff $\exists|\eta\rangle$ that C_A accepts
2. $\lambda \geq b$ if $\nexists|\eta\rangle$ that C_A accepts.

To begin, we define a *history state* as:

$$|\Psi\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} |\phi_t\rangle \otimes |t\rangle_{clock}, \tag{1.6}$$

where the last 'clock' qubits keep track of the 'time' of computation and $|\phi_t\rangle$ represents the state of the circuit at this time t.

Next, we break down H into three components $H = H_I + H_F + H_{prop}$:

$$H_I = \sum_{j=m+1}^{n} |1\rangle\langle 1|_j \otimes |0\rangle\langle 0|_{\text{clock}} \tag{1.7}$$

$$H_F = |0\rangle\langle 0|_1 \otimes |L\rangle\langle L|_{\text{clock}} \tag{1.8}$$

$$H_{\text{prop}} = \frac{1}{2}\sum_t [-U_t \otimes |t\rangle\langle t-1| - U_t^\dagger \otimes |t-1\rangle\langle t| + I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|)] = \sum_t H_t, \tag{1.9}$$

each designed to 'check' a particular condition. If the condition does not hold, a penalty is incurred via an increase in the eigenvalue. For example, H_I checks that the input is of the correct format (last $n-m$ work qubits initialized to $|0\rangle$), and H_F reads out whether or not the input is accepted by C_A (encoded in the first qubit).

$H_{\text{prop},t}$ is a little more complicated, its job is to ensure that state evolution goes according to C_A (backwards evolving terms come from the requirement of Hermiticity). If we have constructed H_{prop} properly, it should incur no penalty acting on a history state:

$$\begin{aligned} H_{\text{prop,t}}|\Psi\rangle &= -\frac{1}{2\sqrt{L+1}}[-U_t|\phi_{t-1}\rangle|t\rangle - U_t^\dagger|\phi_t\rangle|t-1\rangle + |\phi_t\rangle|t\rangle + |\phi_{t-1}\rangle|t-1\rangle] \\ &= -\frac{1}{2\sqrt{L+1}}[-|\phi_t\rangle|t\rangle - |\phi_{t-1}\rangle|t-1\rangle] + |\phi_t\rangle|t\rangle + |\phi_{t-1}\rangle|t-1\rangle]] = 0, \end{aligned} \tag{1.10}$$

so we see that there is indeed no penalty.

Note, we have assumed that the H acts on the subspace including only legal (unary) clock states:

$$|t\rangle_{\text{clock}} = |\underbrace{1,1,...,1}_{t},\underbrace{0,...,0}_{L-t}\rangle \tag{1.11}$$

but we could enforce this explicitly by adding a term to the Hamiltonian, acting only on the clock states, which penalizes 01 patterns:

$$H_{\text{clock}} = \sum_j |0\rangle\langle 0|_j |1\rangle\langle 1|_{j+1}. \tag{1.12}$$

Now consider the case that C_A accepts with probability $\geq 1-\epsilon$ on some input $|\eta, 0\rangle$. Define:

$$|\Omega\rangle = \frac{1}{\sqrt{L+1}}\sum_{t=0}^{L} C_A|\eta,0\rangle \otimes |t\rangle \tag{1.13}$$

then $\langle\eta|H_I|\eta\rangle = \langle\eta|H_{\text{prop}}|\eta\rangle = 0$ and $\langle\eta|H_F|\eta\rangle \leq \frac{\epsilon}{L+1}$, implying that the smallest eigenvalue of H is less than $\frac{\epsilon}{L+1}$ (which we can take as a in our description of LH).

For the second case, where we assume that $\forall|\eta\rangle$, C_A accepts with probability $\leq \epsilon$, we have to consider the ground spaces of $H_1 = H_I + H_F$ and $H_2 = H_{\text{prop}}$ gapped by energies μ_1 and

μ_2, respectively. Then H is gapped by $\lambda_1 - \lambda_0 < \mu = min(\mu_1, \mu_2)$. If we define θ as the angle separating the two closest state vectors between these subspaces ($\theta \neq 0$ due to the incompatibility of H_1 and H_2), it can be shown that $\lambda_{1,\min}(H) \geq 2\mu \sin^2(\theta/2)$ (see, for example ...). Because the history states evolve by a random walk, the mixing time is $\approx n^2$ for a line of length n. Therefore, $\mu = 1/n^2$, $n = L+1$ and $\lambda_{1,\min}(H) \geq \frac{1-\epsilon}{(L+1)^2}$ (which we take as b).

2 Area Laws

A local Hamiltonian is said to be *gapped* if the separation between its ground state and first excited state has some finite value. QMA-complete Hamiltonians are such that $\lambda_1 - \lambda_0 = 1/\text{poly}(n)$. *Area Laws* are a way of bounding the entanglement entropy in the ground states of such Hamiltonians. Roughly speaking, if we spatially divide the Hamiltonian into two pieces an area law says that the entanglement entropy ≤ area of the cut (i.e. the number of terms of the Hamiltonian crossing the cut).