CS 294-2 Fall 2004	Bell States, Bell Inequalities	9/2/04 Lecture 2

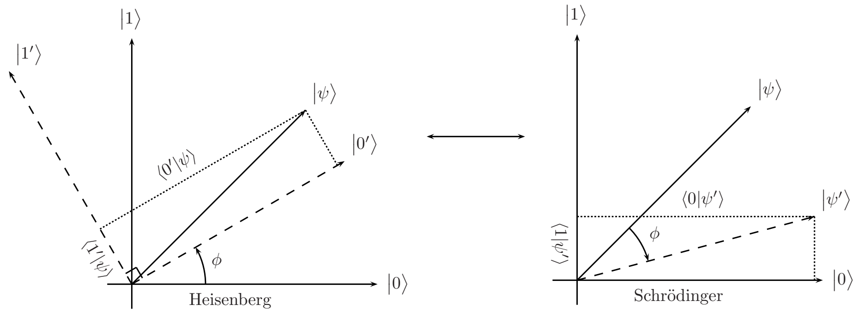
Hilbert Space, Entanglement, Quantum Gates, Bell States, Superdense Coding.

1 One qubit:

Recall that the state of a single qubit can be written as a superposition over its two distinguishable states 0 and 1: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Measuring this qubit in the standard basis, yields outcome 0 with probability $|\alpha|^2$ and resets the state to $|\psi'\rangle = |0\rangle$, and 1 with probability $|\beta|^2$ resetting the state to $|\psi'\rangle = |1\rangle$.

More generally, we can measure the qubit in any orthonormal basis $(|v\rangle, |\perp v\rangle)$. This results in outcome $|v\rangle$ with probability $cos^2\theta$, where θ is the angle between $|\phi\rangle$ and $|v\rangle$, etc. This description corresponds to the Heisenberg picture of quantum mechanics. Here we think of the state of the system as fixed; what changes is the basis in which we measure.

A different viewpoint is provided by the Schrodinger picture. Instead of measuring $|\psi\rangle$ in a rotated basis $(|v\rangle, |\perp v\rangle)$, we achieve the same effect by rotating the entire space, so that $|v\rangle$ is mapped to $|0\rangle$ and $|\perp v\rangle$ is mapped to $|1\rangle$, and then measuring in the standard basis $(|0\rangle, |1\rangle)$.



Such rigid body transformations of the vector space are called unitary transformations. For example, rotations and reflections are unitary. A postulate of quantum physics is that quantum evolution is unitary. That is, if we have some arbitrary quantum system U that takes as input a state $|\phi\rangle$ and outputs a different state $U|\phi\rangle$, then we can describe U as a *unitary linear transformation*, defined as follows.

If U is any linear transformation, the *adjoint* of U, denoted $U^\dagger$, is defined by $(U\vec{v}, \vec{w}) = (\vec{v}, U^\dagger\vec{w})$. In a basis, $U^\dagger$ is the conjugate transpose of U; for example, for an operator on $\mathscr{C}^2$,

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow U^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \quad .$$

We say that U is *unitary* if $U^\dagger = U^{-1}$. For example, rotations and reflections are unitary. Also, the composition of two unitary transformations is also unitary (Proof: U, V unitary, then $(UV)^\dagger = V^\dagger U^\dagger = V^{-1}U^{-1} = (UV)^{-1}$).

Some properies of a unitary transformation U:

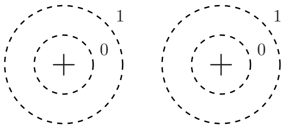
- The rows of U form an orthonormal basis.
- The colums of U form an orthonormal basis.

- U preserves inner products, i.e. $(\vec{v},\vec{w}) = (U\vec{v},U\vec{w})$. Indeed, $(U\vec{v},U\vec{w}) = (U|v\rangle)^{\dagger}U|w\rangle = \langle v|U^{\dagger}U|w\rangle = \langle v|w\rangle$. Therefore, U preserves norms and angles (up to sign).
- The eigenvalues of U are all of the form $e^{i\theta}$ (since U is length-preserving, i.e., $(\vec{v},\vec{v}) = (U\vec{v},U\vec{v})$).
- U can be diagonalized into the form

$$\begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{i\theta_d} \end{pmatrix}$$

2 Two qubits:

Now let us return to the case of two qubits. Consider the two electrons in two hydrogen atoms:



Recall that the quantum state of these two electrons is a superposition of the four classical states, 00, 01, 10 and 11:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \quad ,$$

where $\sum_{ij}|\alpha_{ij}|^2 = 1$. Again, this is just Dirac notation for the unit vector in $\mathscr{C}^4$:

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

where $\alpha_{ij} \in \mathscr{C}$, $\sum |\alpha_{ij}|^2 = 1$.

Tensor products (informal):

Recall that the state of each qubit is an element of the Hilbert space C^2. The state of a two qubit system is an element of the Hilbert space C^4. How do we glue together the two copies of C^2 to get the composite Hilbert space C^4? This is via an operation called tensor product. It works as follows: suppose the state of the first qubit is $|\phi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \in \mathscr{C}^2$, and the state of the second qubit is $|\phi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle \in C^2$. Then their joint state is described by the tensor product, which can be described informally as follows:

$$\begin{aligned} |\phi\rangle &= |\phi_1\rangle \otimes |\phi_2\rangle \\ &= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle \quad . \end{aligned}$$

This operation will be described more formally in the next lecture. For now, we just note that states of the two qubits where we can specify the state of each qubit individually are very special - they are called tensor product states. The typical state of a two qubit system is not a tensor product state, and is said to be entangled. The Bell state introduced in the last lecture is an example of a highly entangled state.

3 Examples of Unitary transformations

Hadamard gate:

The Hadamard gate may be viewed as a reflection about the line $\theta = \pi/8$, or as a rotation through $\pi/4$ followed by a reflection about the line $\theta = \pi/4$. This transformation maps the state $|0\rangle$ to $|+\rangle$ and the state $|1\rangle$ to $|-\rangle$:

$$|0\rangle \xrightarrow{H} \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle \equiv |+\rangle \tag{1}$$

$$|1\rangle \xrightarrow{H} \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle \equiv |-\rangle \ . \tag{2}$$

In matrix form, we write

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ .$$

Note that $H^\dagger = H$ – since H is real and symmetric – and $H^2 = I$. Our notation $|+\rangle = H|0\rangle$ and $|-\rangle = H|1\rangle$ emphasizes the fact that the information is encoded in the phase — measuring the state $|+\rangle$ or $|-\rangle$ in the standard basis yields 0 and 1 with equal probability. The Hadamard transform thus transforms bit information into phase information and vice versa. This is a fundamental property that will be extensively used.

In a quantum circuit diagram, we imagine the qubit traveling from left to right along the wire. The following diagram shows the application of a Hadamard gate.

NOT gate: The not gate (denoted by X) swaps the bases vectors of the basis 0 and 1. It maps 0 to 1 and vice-versa. Thus, by linearity, it maps $\alpha|0\rangle + \beta|1\rangle$ to $\alpha|1\rangle + \beta|0\rangle$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

PHASE-FLIP gate: The phase flip gate (denoted by Z) applies a phase of -1 to $|1\rangle$ and leaves $|0\rangle$ unchanged.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} Z|0\rangle &= |0\rangle \\ Z|1\rangle &= |-1\rangle \\ Z|+\rangle &= |-\rangle \\ Z|-\rangle &= |+\rangle \end{aligned}$$

The last two equations suggest that in the Hadamard basis, the phase flip gate acts as a Not gate. i.e. Z=HXH.

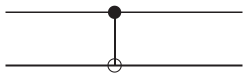
CNOT gate: The controlled-not (CNOT) gate exors the first qubit into the second qubit ($\left|a,b\right\rangle \to \left|a,a\oplus b\right\rangle = \left|a,a+b \bmod 2\right\rangle$). Thus it permutes the four basis states as follows:

$$\begin{array}{ll} 00 \to 00 & 01 \to 01 \\ 10 \to 11 & 11 \to 10 \ . \end{array}$$

As a unitary 4×4 matrix, the CNOT gate is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In a quantum circuit diagram, the CNOT gate has the following representation. The upper wire is called the control bit, and the lower wire the target bit.



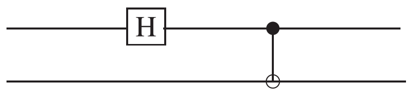
Bell states:

The four Bell states are:

$$\begin{aligned} \left|\Phi^{\pm}\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle \pm \left|11\right\rangle\right) \\ \left|\Psi^{\pm}\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle \pm \left|10\right\rangle\right) \ . \end{aligned}$$

These are maximally entangled states on two qubits.

We can generate the Bell states with a Hadamard gate and a CNOT gate. Consider the following diagram:



The first qubit is passed through a Hadamard gate and then both qubits are entangled by a CNOT gate.

If the input to the system is $|0\rangle \otimes |0\rangle$, then the Hadamard gate changes the state to

$$\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle \ ,$$

and after the CNOT gate the state becomes $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the Bell state $|\Phi^+\rangle$. In fact, one can verify that for each of the four standard basis states as inputs, the output is the the corresponding Bell basis state:

$$|00\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle \tag{3}$$

$$|10\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle \tag{4}$$

$$|01\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle \tag{5}$$

$$|11\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi^-\rangle \tag{6}$$

(7)

The fact that the quantum circuit carries out a unitary transform implies that the Bell basis states form an orthonormal basis for C^4. Indeed, the reverse of this circuit (feeding the input at the output wires and reading out the output at the input) transforms the Bell basis states into the standard basis — measuring the output in the standard basis thus implements a Bell basis measurement.

4 Super dense Coding

Consider a situation where Alice receives 2 classical bits and wishes to communicate them to Bob. Clearly she must send him at least 2 classical bits to convey her information to him. If Alice and Bob share a Bell state, then Alice can convey her two classical bits by sending him a single quantum bit. In the next lecture we shall show that this is optimal - it is impossible to transmit more than two classical bits by sending 1 qubit.

Alice and Bob share a Bell state Φ^+. Suppose Alice receives the classical bits x_1 and x_2. Alice applies one of four gates to her qubit of the Bell state depending upon x_1x_2 to convert the Bell state to one of the four Bell basis states. Now if Alice sends her qubit to Bob, he can read off which of the four Bell basis states he has by performing a Bell basis measurement:

- If $x_1x_2 = 00$ then she sends her qubit unchanged.
- If $x_1x_2 = 10$ then she applies the phase-flip gate Z to her qubit.
- If $x_1x_2 = 01$ then she applies the Not gate X to her qubit.
- If $x_1x_2 = 1$ then she applies both the phase-flip and the NOT gate ZX to her qubit.

Upon receiving Alice's part of the Bell state, Bob runs the circuit described in the previous section in reverse to obtain two qubits y_1y_2. He then measures this state in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In other words, he applies the CNOT gate to b and to his quantum bit. He then applies the Hadamard to b (the quantum bit he received from Alice) and thus, retrieves x_1x_2.