Quantum Fourier Transform (QFT)

Quantum Computation is all about QFT. Why? Need unitary transformations that can be decomposed efficiently (MEANING?). There only a few types of such non-trivial transformations, and FT is an important class of them.

FT lie in the heart of mathematical apparatus of quantum mechanics.

We will discuss Fourier transforms over group $\mathbb{Z}_m$. Note $\omega = e^{\frac{2\pi i}{m}}$.

Classically, Fourier Transform is given by the following equation:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & \ldots & \omega^{jk} & \\
1 & \ldots & \omega^{(m-1)(m-1)} & \\
1 & \ldots & \ldots & \\
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\vdots \\
\alpha_{m-1} \\
\end{pmatrix}
=
\begin{pmatrix}
\hat{\alpha}_0 \\
\vdots \\
\hat{\alpha}_{m-1} \\
\end{pmatrix},
$$

where $j,k = 0..m-1$.

Normally matrix-vector multiply involves $O(m^2)$ operations. However, exploiting the structure of the Vandermonde matrix, we can cut the complexity to $O(m \log m)$. The algorithm is called Fast Fourier Transform and is discussed below.

Quantumly, the $m$-dimensional complex vector is the state of $\log m$ qubits. The QFT is the mapping

$$
\sum_{j=0}^{m-1} \alpha_j \left| j \right> \xrightarrow{\text{QFT}} \sum_{j=0}^{m-1} \hat{\alpha}_j \left| j \right>
$$

Advantage: QFT can be realized with a circuit of size $O(\log^{O(1)} m)$, a speedup over FFT’s $O(m \log m)$.

Disadvantage: Coefficients $\hat{\alpha}_j$ are quantum states and thus are not readily accessible. All we get is Fourier Sampling. If we measure, we get $\left| j \right>$ with probability $|\hat{\alpha}_j|^2$. 
Building QFT circuit of polylog size
Describe QFT’s divide and conquer idea here

Properties of Quantum Fourier Transforms

1. **F is unitary:** \( FF^\dagger = F^\dagger F = I \).
   Proof. Note that \((j,k)^{th}\) entry of \( FF^\dagger \) is
   \[
   \frac{1}{m} \sum_i \omega^{ij} \omega^{-ik} = \frac{1}{m} \sum_i \omega^{(i-j)k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \text{ (geometric series with } a = \omega^{i-k} \text{)}
   
   sums to \( \frac{a^m - 1}{a - 1} = \frac{1 - 1}{a - 1} = 0 \), since \( \omega = \sqrt{1} \).

2. **Pick out similarities**
   Consider \( |\phi\rangle = \sum_j \alpha_j |j\rangle \) and \( |\phi^\dagger\rangle = \sum_j \alpha^*_j |j + 1\rangle \). How do \( |\phi\rangle \) and \( |\phi^\dagger\rangle \) differ? Of course, there could be no overlap. So it’s hard for us to know that \( \phi \approx \phi^\dagger \).

   Consider the respective QFTs:
   \[
   |\phi\rangle \rightarrow \sum_j \hat{\alpha}_j |j\rangle \text{ and } |\phi^\dagger\rangle \rightarrow \sum_j \hat{\beta}_j |j\rangle .
   
   The beauty of QFT is that \( |\hat{\alpha}_j| \approx |\hat{\beta}_j| \)!

3. **Pick out symmetries**
   talk how FT is concentrated for uniform initial superposition and how it can be used for period finding.