# When are Best Fit and First Fit Optimal?
C. C. McGeoch & J. D. Tygar
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# When are Best Fit and First Fit Optimal?

C. C. McGeoch[*]          J. D. Tygar[†]

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# 1  Abstract

We examine on-line heuristics for one-dimensional bin-packing when $n$ weights are chosen uniformly over the interval $(\ell, u)$, $0 \le \ell \le u \le 1$. Bentley et al. [2] have shown that the First Fit heuristic is asymptotically optimal for weights chosen uniformly from the real interval $(0, 1)$. Shor [13] has given a similar result for the Best Fit heuristic.

We show that expected empty space in First Fit packings of weights drawn uniformly from the real interval $(0, .85)$ is $\Omega(n)$. This implies that First Fit is not optimal in the expected case for this input distribution. The analysis extends to show First Fit is also not optimal when $u$ lies in some neighborhood around .85.

We also show that expected empty space is $\Omega(n)$ for any $\ell \neq .15$ when $u = .85$. This implies that First Fit is not optimal for an infinite set of $\ell$ values.

We extend these bounds to hold for the Best Fit heuristic over the same input distributions.

Finally, we present experimental results characterizing First Fit for weight lists of the form $(0, u)$ for other upper bounds $u$.

## 2  Introduction

The *one-dimensional bin packing problem* is to pack a list of $n$ items with weights from $[0, 1]$ into a minimum number of unit-capacity bins. This problem is known to be NP-hard (see [5]), and many heuristic algorithms have been proposed. The well-known *First Fit* heuristic inspects the bins sequentially and places each item into the first (oldest) bin that can contain it. The items cannot be reordered—that is, they must be packed in the order in which they are presented as input. First Fit can be implemented to run in $O(n \log n)$ time by imposing a heap data structure over the bins. The *Best Fit* heuristic places items into bins, such that the $i^{th}$ item of weight $w_i$ is placed into the bin which has the greatest total weight less than than $1 - w_i$.

A standard expected-case model assumes that input to a packing heuristic is a list of weights $L_{\ell,u,n} = < w_1, w_2, \ldots, w_n >$ drawn independently and uniformly from the interval $(\ell, u)$, for $0 \leq \ell < u \leq 1$. We view $L_{\ell,u,n}$ as a random variable with parameters $\ell$, $u$, and $n$, and are interested in the expected values of functions on this random variable. Let $B_{FF}(L_{\ell,u,n})$ be the expected number of bins required to pack such a list by the First Fit heuristic. The expected number of bins required in the optimal packing of the list is

denoted by $B_{OPT}(L_{\ell,u,n})$. The *bin ratio* is the asymptotic ratio $\lim_{n\to\infty} \sup B_{FF}(L_{\ell,u,n})/B_{OPT}(L_{\ell,u,n})$. The bin ratio is always greater than or equal to 1; informally, smaller bin ratios correspond to better heuristics. The *empty space* in a First Fit packing—the total amount of space left at the top of partially-filled bins—is denoted by $ES_{FF}(L_{\ell,u,n})$ and equal to $B_{FF}(L_{\ell,u,n}) - \sum_{i=1}^{n} w_i$.

A bin-packing heuristic produces optimal behavior in an expected-case model when the bin ratio approaches 1. It sufficient to prove that expected empty space is $o(n)$, since the optimal packing must use $\Omega(n)$ bins (because it must contain the entire list, which has expected total weight $(u-\ell)n/2$). Bentley et al. [2] have shown that when $\ell = 0$ and $u = 1$, expected empty space is $O(n^{4/5})$, which implies that First Fit is asymptotically optimal. Their result can be extended to any distribution symmetric around .5. Shor [13] improved the bound on empty space for $\ell = 0$ and $u = 1$ to $O((n \log n)^{2/3})$ and $\Omega(n^{2/3})$. His result can be extended easily to any uniform distribution such that $\ell = 1 - u$ and $u \geq 2/3$. He also showed that empty space for Best Fit has bounds of $O(n^{1/2} \log n)$ and $\Omega(n^{1/2}(\log n)^{3/4})$.

Karmarkar [8] has shown that when $\ell = 0$ any $u$ admits a *perfect packing*, that is, for the optimal packing rule OPT, we have $\lim_{n\to\infty}(B_{OPT}(L_{0,u,n})/ \sum_{i=1}^{n} w_i) \to 1$. Lueker [10] showed that perfect packings occur for uniform distributions on $(\ell, u)$ whenever the interval $(\ell, u)$ can be partitioned into units each symmetric around $1/k$, for a collection of integers $k > 1$. For example, the interval $(.15, .75)$ admits a perfect packing because it can be partitioned into $(.15, .25)$ and $(.25, .75)$, which are symmetric around $1/5$ and $1/2$, respectively. He also showed that perfect packings cannot occur for certain other values of $\ell$ and $u$. To demonstrate non-optimality of a bin packing heuristic when a perfect packing is admitted—i.e.,

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that the bin ratio approaches some constant strictly greater than 1—it suffices to show that expected empty space for the heuristic is $\Omega(n)$.

The results by Bentley et al. [2] and Shor [13] are the only expected-case analyses of First Fit and Best Fit to date. Others have considered worst-case performance. Johnson [6] derived a tight worst-case upper bound of 1.7 on the bin ratio for the two online heuristics. (An online heuristic must pack each item as it arrives, with no reordering.) Brown [3]_has shown that there must exist lists such that any online heuristic (including First Fit and Best Fit) gives a packing ratio of at least 1.536. For an extensive survey of bin packing heuristics, see Coffman, Garey and Johnson [4].

A few experimental studies have appeared that examine packing performance at $\ell = 0$ and other values of $u$. Johnson [6] studied many heuristics and estimated the bin ratio for values of $n$ up to 200 and for $u = .25, .5$, and 1. Ong, Magazine, and Wee [12] estimated the bin ratio for four heuristics using values of $n$ up to 1000.

The conjecture that First Fit and Best Fit are asymptotically optimal when $\ell = 0$ and $u = 1$ was first made as a result of an experimental study of four bin packing heuristics reported by Bentley et al. [1]; this conjecture was subsequently proved in Bentley et al. [2]. It was also conjectured that there exist values of $u$ near .84 for which the heuristics leave at least linear empty space and therefore is not asymptotically optimal. Several observations and conjectures relating to bin packing are reported by Bentley et al. [1] and McGeoch [11]; the experiments described there have been extended and new results are presented in Section 4 of this paper.

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We show in the following section that $ES_{FF}(L_{\ell,.85,n}) = \Omega(n)$ whenever $\ell \neq .15$. In particular, this implies that First Fit is not optimal in the expected case when $\ell = 0$ and $u = .85$, although the heuristic is optimal when $\ell = 0$ and $u = 1$. Put another way, First Fit packings of weights from $(0, .85)$ leave much more empty space per bin than packings of weights from $(0, 1)$. We extend this proof for First Fit to show that Best Fit also fails to be asymptotically optimal on the distribution $(0, .85)$

## 3  Lower Bounds for $u = .85$

We first analyze the expected amount of empty space in First Fit packings of $n$ weights drawn uniformly from $(0, .85)$. We later show how this can be extended to Best Fit. We begin by defining some terms and giving an informal sketch of the proof.

A bin containing $k$ items is called a *k-bin*. The *gap* in a bin is the capacity remaining in the bin; empty space is simply the sum of gaps in all bins. The *total weight* of a bin is the sum of the weights in that bin, so the total weight of a bin plus its gap equals 1.

The *candidate sequence* in a First Fit packing is defined as follows. The first bin of the complete sequence of bins is in the candidate sequence. Let the $i + 1^{st}$ bin of the candidate sequence be the first (oldest) bin in the complete sequence of bins satisfying the following.

1. Its total weight is less than that of the $i^{th}$ bin of the candidate sequence.

2. It occurs later in the complete bin sequence than the $i^{th}$ bin of the candidate sequence.

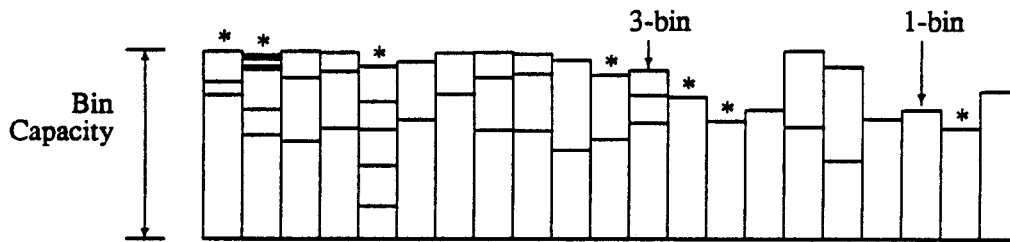The bins in the candidate sequence are the only bins which can accept new items under the First Fit

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Figure 1: First Fit packing of 50 weights from (0, .85). Candidate bins are marked with stars.

heuristic, for if an item is small enough to fit in a non-candidate bin, it will be trapped in an earlier

candidate bin. Figure 1 shows a First Fit packing of fifty weights drawn from (0, .85). The candidate

bins in this packing are marked with stars. A 1-bin and a 3-bin are also marked.

Note that the bins in the candidate sequence form a "filter" on new items arriving in the input list.

If two bins adjacent in the candidate sequence have a difference of $\epsilon$ in their gaps and a new weight is

placed in the second bin, then the resulting gap in the second bin must be at most $\epsilon$. If the candidate

sequence is long and smooth, newly-arriving weights will tend to be packed tightly.

Our analysis formalizes the intuition that 2-bins tend to have smaller gaps than 1-bins. Therefore 2-

bins in the candidate sequence tend to occur *before* 1-bins in the candidate sequence. We show, essentially,

that a discontinuity appears consistently between 2-bins and 1-bins and that the candidate sequence is

not smooth here. Because the 1-bins do not benefit from a smooth candidate sequence, they tend to be

matched with unsuitable new weights at a rate high enough to leave linear empty space in the packing.

Let $\alpha$ equal the total weight of the bin in the candidate sequence having largest gap less than .15.

Equivalently, $\alpha$ is the smallest value greater than .85 such that there is a bin in the candidate region with total weight $\alpha$. We will analyze the rate at which 1-bins pair with new weights to form 2-bins, for arbitrary $\alpha$. Note that the rate at which candidate bins of total weight $a$ are formed is bounded above by the rate at which all bins with total weight $a$ are formed. Furthermore, by the nature of First Fit packings, if there exists a bin with enough capacity to hold an incoming item, then there must exist a bin in the candidate sequence (perhaps the same bin) with enough capacity to hold the item. Every bin that is paired will eventually show up in the candidate sequence with probability 1. To analyze the formation rate of candidate bins, therefore, we need only consider the formation rate for all bins.

Theorem 3.2 shows that there exists a *balance point* $\alpha_0$, strictly less than 1 and strictly greater than .85, such that: (1) candidate bins having total weight less $\alpha_0$ are created at a slower rate than items arrive in the input list to match with them; and (2) candidate bins having total weight greater than the balance point are created faster than new items arrive to match with them. Property 1 means that candidate bins with total weight in the range $(.85, \alpha_0)$ appear rarely, so the filter tends to have a discontinuity of size at least $\alpha_0 - .85$, causing 1-bins to be packed badly. Property 2 means that there aren't enough small items available to "fix" the badly-packed bins.

As a consequence of Theorem 3.2, there must exist a $\beta \in (\alpha_0, 1)$ such that bins with gap at least $1 - \beta$ are formed more quickly than they are removed. Therefore a constant proportion of bins have at least constant gap at any time during the packing. This immediately give the linear lower bound on empty space, stated formally in Theorem 3.3.

In order to analyze formation rates for 2-bins we define a function $D_\alpha(x, y)$, with domain $.5 \le x < .85$, for $1 - \alpha < y < .5$, and for $.65 \le x + y \le 1$. If 1-bins in the candidate sequence were generated with uniform distribution and form an arbitrarily fine filter, then this function would give the rate at which a small weight $y$ of size in the range $(1 - \alpha, .5)$ is paired with a 1-bin having total weight $x$ at least $.5$ (and, necessarily, at most $.85$). In fact, First Fit does not generate an arbitrarily fine filter. However, a coarse filter can only increase the amount of expected empty space. (We will return to this point below). Thus this analysis will give a lower bound on the total amount of empty space.

We remark that although $D_\alpha(x, y)$ does not capture all possible pairing situations—for example, it does not consider 1-bins with total weight less than $.5$—it is sufficient to give the desired bounds, as we show below.

Let $P_\alpha(z)$ represent the rate at which bins of total weight $z$ are formed by pairing: that is $P_\alpha(z) = \int_x D_\alpha(x, z - x) \, dx$, for $x$ and $y = z - x$ in the ranges defined above for $D_\alpha(\cdot)$. Figure 2 presents a diagram of the domain over which $D_\alpha(x, y)$ and $P_\alpha(z)$ are defined. $D_\alpha(x, y)$ is defined for each coordinate pair $(x, y)$ inside the region. $P_\alpha(z)$ corresponds to a diagonal line such that $z = x + y$. The subregions drawn in the interior of the diagram will be used in the proof of Theorem 2.2.

Let $\theta(z)$ be the rate at which bins with total weight less than $z$ form by all methods (not just pairing). Let $\phi(z) = d\theta(z)/dz$ be the instantaneous rate at which bins of size $z$ are formed. Let $r$ be the rate at which items arrive in the input list of size $z$; for a uniform distribution on weights, this is a constant. Finally, for notational convenience, let $w$ denote the difference $\alpha - .85$.

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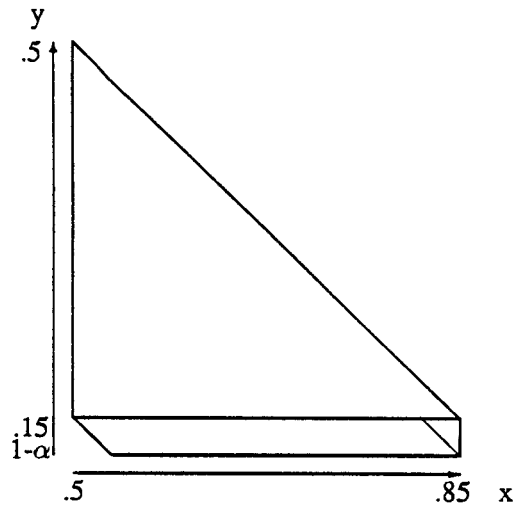
Figure 2: Pairing Functions

We first state the following useful lemma.

**Lemma 3.1** *For $(x, y)$ having ranges defined above,  $D_\alpha(x, y) = c_y e^{x/w}$.*

**Proof:** It is easy to see that the rate of change of $D_\alpha(x, y)$ with respect to $x$ is an exponential function depending upon $w = \alpha - .85$, the size of the discontinuity in the filter. That is,

$$\frac{\partial D_\alpha(x, y)}{\partial x} = \frac{D_\alpha(x, y)}{w}.$$

Solution of this partial differential equation gives the desired result. The coefficient $c_y$ depends upon the value of $y$. $\square$

The following theorem shows the existence of the balance point $\alpha_0$.

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**Theorem 3.2** *There exists an $\alpha_0 \in (.85, 1)$ such that,*

$$\phi(z) < r \quad \forall z \in [.85, \alpha_0), \text{ and}$$

$$\phi(z) > r \qquad \forall z \in [\alpha_0, 1]. \tag{1}$$

**Proof:** We first show that the there exists an $\alpha_p$ such that the above inequalities hold when $P_{\alpha_p}(z)$, the rate bins are formed by pairing, is substituted for $\phi(a)$, the rate bins are formed by all methods. We then show that $\alpha_0$ must exist for the latter case.

By Lemma 2.1, we have $D_\alpha(x, y) = c_y e^{x/w}$. We must find appropriate values for $c_y$ and then derive $P_\alpha(z)$.

First consider the case $y \in (1 - \alpha, .15]$. It is easy to see that $D_\alpha(.85, y) = r/w$. Because by the definition of $\alpha$ there are no candidate bins of size in the range $(.85, \alpha)$ weights of size $y \in (1 - \alpha, .15]$ are paired with 1-bins of size .85 at the total pairing rate for these bins divided by the size of the discontinuity. Also, $c_y$ must be a constant for $y$ in this range since the weights arrive according to a uniform distribution and none are filtered in the candidate sequence. Combining these facts with Lemma 2.1, we have for $y \in (1 - \alpha, .15]$,

$$c_y e^{(.85/w)} = \frac{r}{w}, \text{ or}$$

$$c_y = \frac{r}{w} e^{(-.85/w)}.$$

Now consider the case $y \in (.15, .5]$. Here we have $D_\alpha(1 - y, y) = r/w$. That is, weights that perfectly

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match with existing 1-bins arrive at a uniform rate and are not filtered in earlier 1-bins, but they must compete with smaller weights not caught by earlier 1-bins to a degree determined by $w$. We also know that $D_\alpha(x, y) = D_\alpha(x + y - .15, .15)$ for $y > .15$. Combining this with Lemma 2.1 gives

$$c_y = \frac{r}{w} e^{(y-1)/w}.$$

Therefore, for the defined domain over $x$ and $y$ we have

$$D_\alpha(x, y) = \frac{r}{w} e^{(x/w)} \cdot \begin{cases} e^{-.85/w} & y \in (1 - \alpha, .15] \\ e^{(y-1)/w} & y \in (.15, .5] \end{cases}$$

We must now find the rate $P_\alpha(z)$ at which bins of total weight $z = x + y$ are formed by pairing. We first consider the small triangular region at the bottom right corner of Figure 2, defined by $y \in (1 - \alpha, .15]$ and $z \in [.85 + (1 - \alpha), 1]$. For this area the pairing rate is

$$P_\alpha(z) = \int_{z-.85}^{.15} \frac{r}{w} e^{(z-y)/w} e^{-.85/w} \, dy,$$

which gives

$$P_\alpha(z) = r e^{(z-1)/w} \left[ e^{(1-z)/w} - 1 \right].$$

The small strip at the bottom of Figure 2 is defined by $y \in (1 - \alpha, .15]$ and $z \in [.65, .85 + (1 - \alpha))$. For this area, we have

$$P_\alpha(z) = \int_{.15-w}^{.15} \frac{r}{w} e^{(z-y)/w} e^{-.85/w} \, dy,$$

or

$$P_\alpha(z) = r e^{(z-1)/w} [e - 1].$$

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Finally, for $y \in (.15, .5]$, the large triangular region of of Figure 2, the pairing rate for any $z \in [.65, 1]$ is given by

$$P_\alpha(z) = \frac{r\sqrt{2}}{w}(z - .65)e^{(z-1)/w}.$$

Consolidating these formulas and substituting $\alpha - .85$ back for $w$, we have the rate at which 2-bins of total weight $z \in [.65, .85 + (1 - \alpha)]$ are formed by pairing is

$$P_\alpha(z) = re^{(z-1)/(\alpha-.85)}\left[\frac{\sqrt{2}(z - .65)}{\alpha - .85} + e - 1\right],$$

and the rate for $z \in (.85 + (1 - \alpha), 1]$ is

$$P_\alpha(z) = re^{(z-1)/(\alpha-.85)}\left[\frac{\sqrt{2}(z - .65)}{\alpha - .85} + e^{(1-z)/(\alpha-.85)} - 1\right].$$

The variable $r$ corresponds to the rate at which 1-bins of total weight .85 are paired. Clearly the pairing rate for these bins is equivalent to the instantaneous rate at which items of size .85 arrive in the input list: the pairing cannot be greater than the arrival rate, and there are more items available to pair with 1-bins of total weight .85 than there are items of this size, so they are all paired.

Let $z_\alpha$ be the value of $z$ for which $P_\alpha(z) = r$. Inspection of $P_\alpha(z)$ reveals that such a $z_\alpha$ exists for any $\alpha \in (.85, .948)$ (for $\alpha > .948$, $P_\alpha(z) > r, \forall z$). Furthermore, $z_\alpha$ decreases as $\alpha$ increases, so there must exist an $\alpha_p \in (.85, .948)$ such that $P_{\alpha_p}(\alpha_p) = r$. We observe that $\alpha_p \approx .8966$: if First Fit were modified so that only 2-bins with total weight at least .85 and 1-bins with total weight at least .5 appeared in the candidate sequence, then 2-bins with total weight in the range $(.85, .8966]$ would form more slowly than new items are available to match with them, and 2-bins with weight in the range $(.8966, 1]$ would form more quickly than small items arrive to match with them.

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Now let $\phi_\alpha(z)$ replace $P_\alpha(z)$. This function represents the formation rate from all methods, including

pairing. We must show that an $\alpha_0$ corresponding to $\alpha_p$ exists that is strictly in the range $(.85, 1)$. Note

that $\alpha_p \geq \alpha_0$, since $\forall z, \alpha(\phi_\alpha(z) \geq P(z))$, so we need only show that $\alpha_0 > .85$. First, our analysis does not

consider 1-bins with total weight below $.5$. There can be at most one bin in a First Fit packing with total

weight less than $.5$, and it must be the last bin in the candidate sequence; therefore the small 1-bins by

themselves cannot affect pairings in the candidate sequence. However we must consider 2-bins formed

later from pairing these small 1-bins (we call these *illegitimate* 2-bins) and their effect on the pairing rate.

An illegitimate 2-bin of size $> .85$ could fall into the range $(.85, \alpha)$, increasing the formation rate of

these bins and possibly lowering the balance point. Inspection of the arrival rates reveals that a balance

point greater than $.85$ must still exist, since illegitimate 2-bins of this size form very rarely.

Illegitimate 2-bins of size $\leq .85$ can also form. In addition, legitimate 2-bins of sizes $\leq .85$ can

also form outside of the range of the $D$ function when, for example, a $.12$ weight pairs with a $.5$ weight.

Inclusion of these bins in the candidate sequence have the effect of changing a uniform distribution

(composed of 1-bins) to a "top-heavy" distribution. This can only increase the total amount of expected

empty space.

We must also show that $\alpha$ approaches $\alpha_0$ with probability 1 as $n \to \infty$. The proof of this is a

straightforward application of classical renewal theory (see [7]). Intuitively, when $\alpha < \alpha_0$, future values

of $\alpha$ are more likely than not to be higher than the current value of $\alpha$, since in this case $z_\alpha > \alpha_0$.

Similarly, when $\alpha > \alpha_0$ we have $z_\alpha < \alpha_0$, so new bins with total weight slightly less than $\alpha_0$ tend to be

created, pushing the next $\alpha$ down towards $\alpha_0$. $\square$

**Theorem 3.3** *Empty space in a First Fit packing of weights drawn uniformly from* $(0, .85)$ *is* $\Omega(n)$, *with probability approaching 1 as* $n \to \infty$.

**Proof:** Select a $\beta$ between $\alpha_0$ and 1. This is always possible because $\alpha_0$ is strictly less than 1. Bins with total weight in the range $(\alpha_0, \beta)$ are formed more quickly than they are removed. Therefore a constant proportion of these bins are present in the packing at any time. These bins have at least constant gap. There must be at least a linear number of bins in the packing, because even the optimal packing must leave at least a linear number of bins. Since a constant proportion of these bins have at least constant gap, there must be at least a linear amount of empty space in the packing.

We observe that a coarse filter increases the expected empty space in 2-bins. Inspection of Theorem 2.2 demonstrates that with a coarse filter bins what fall in the range $(\alpha, \beta)$, where $\beta \in (\alpha, 1)$, must be created more quickly than they would be with an arbitrarily fine filter and hence more quickly than they can be filled, so the theorem continues to hold. $\square$

Suppose now that the lower bound on the distribution of weights is nonzero. Our analysis allows characterization of empty space in First Fit packing of weights drawn uniformly from $(\ell, .85)$.

**Theorem 3.4** *Empty space in First Fit packings of weights drawn uniformly from* $(\ell, .85)$ *is* $\Omega(n)$ *whenever* $\ell \neq .15$, *with probability approaching 1 as* $n \to \infty$,

**Proof:** Suppose $0 < \ell < (1 - \alpha_0)$ for the balance point $\alpha_0$ described above. Then even fewer small

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items arrive in the input list to fix badly-packed bins (with total weight in $(\alpha_0, \beta)$). The linear lower bound certainly continues to hold.

Suppose $(1 - \alpha_0) \leq \ell < .15$. In this case, items larger than $\ell$ but smaller than $1 - \alpha_0$ will not be matched with 1-bins. This imposes an artificial upper bound on the weights arriving to match with 1-bins, identical in effect to imposing the restriction $\alpha \leq 1 - \ell < \alpha_0$ on the filter. But a $z_\alpha$ strictly less than 1 must exist for any $\alpha < \alpha_0$, and so the desired $\beta$ must exist. Therefore the linear lower bound still holds.

Suppose $\ell > .15$. Then First Fit as well as the optimal algorithm must leave $\Omega(n)$ empty space. Two weights larger than $1/2$ cannot share a bin, so every weight larger than $1/2$ must have its own bin. If the number of weights larger than $1/2$ is greater than the number of weights smaller than $1/2$, then a constant proportion of the large weights cannot be paired with small weights. A linear lower bound on empty space is immediate. $\square$

Analysis of Best Fit requires only the observation that the entire set of bins forms the filter rather than the candidate sequence of bins. Hence we immediately have this theorem:

**Theorem 3.5** *Empty space in Best Fit packings of weights drawn uniformly from $(\ell, .85)$ is $\Omega(n)$ whenever $\ell \neq .15$, with probability approaching 1 as $n \to \infty$,*

When are First Fit and Best Fit optimal for weights drawn from $(\ell, .85)$? If the heuristics as well as the optimal algorithm leaves $o(n)$ empty space, then they are asymptotically optimal. This is the case for weights drawn from $(.15, .85)$.

The heuristics are not optimal when it leaves $\Omega(n)$ empty space and the optimal algorithm allows

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a perfect packing. Lueker [10] characterized the values of $(\ell, u)$ for which perfect packings occur. When $u = .85$, the $\ell$ values giving sublinear empty space are those such that the distribution of weights can be broken into intervals each symmetric around $1/k$ for appropriate integers $k$. For example, $\ell = 1/6 - (.15 - 1/6) \approx .1357$ allows a perfect packing because the distribution can be broken into intervals $(.15, .85)$ and $(.1357, .15)$, which are symmetric around $1/2$ and $1/6$, respectively.

When the heuristics as well as the optimal algorithm leave linear space, the question of heuristic optimality depends upon the constants implicit in the $\Omega(n)$. Further analysis is needed here.

An even more interesting question, of course, is what happens when we vary $u$? Our derivation of $P_\alpha(z)$ can be easily extended to any $u$ near $.85$, but we require a deeper understanding of secondary effects in order to extend our analysis beyond a small region of $u$ values near $.85$. The next section considers this question further, for First Fit.

# 4 Experimental Results

A long-standing open problem has been to characterize First Fit for $\ell = 0$ and any $u$. In particular, for what values is expected empty space $\Omega(n)$ and for what values is it $o(n)$ ? Certainly the analysis of the previous section could be modified to hold for $u$ in a small range near $.85$, but our technique is not exact enough for a precise characterization of the $u$ values giving non-optimal packings. This section presents experimental results describing First Fit packings for other values of $u$.

The simulation program was implemented in C on a VAX/780 operating under Berkeley Unix 4.3[1].
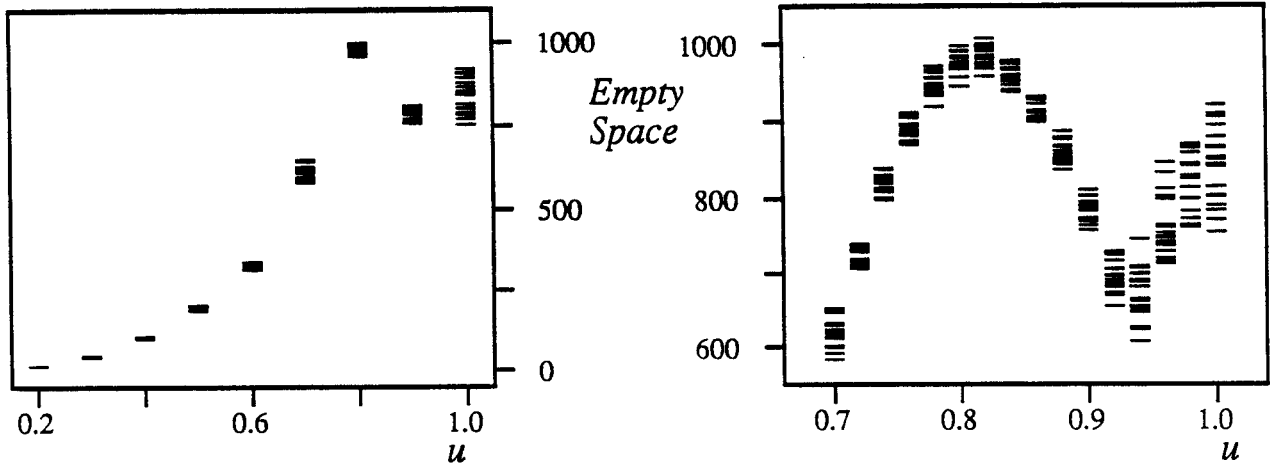
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Figure 3: These graphs depict empty space with respect to $u$ for $n = 128,000$ and 20 trials at each sample point.

Weights were represented by 32-bit integers, with $2^{23} - 1$ representing the bin capacity. The weights were generated by Marsaglia's cyclic feedback method as described by Knuth [9] (Algorithm A, Section 3.2.2, Second Edition). In the experiments, $n$ was set at $250, 500, \ldots 128000$. The simulation program could generate and pack 128000 weights in about 60 seconds; the experiments described here required about 10 hours of CPU time.

The graphs in Figure 3 depict empty space as a function of $u$ for values of $u$ ranging from 0.2 to 1 and from 0.7 to 1.0. Twenty trials were taken at each $u$ for $n$ set at $= 128,000$. These and related results (see [1] and [11]) prompted the original conjecture that although empty space grows as $o(n)$ when $u = 1$, empty space grows linearly for some values of $u$ near .84. This conjecture was the starting point for our analysis and motivated our choice of .85 as an interesting upper bound on weight sizes.

An obvious approach to characterizing the asymptotic behavior of First Fit for other values of $u$ is to measure growth in $n$ at fixed $u$. This approach is not entirely satisfying, however, because we cannot

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measure asymptotic behavior experimentally. At $u = 1$, for example, standard model-fitting analyses of samples taken at $n = 1000, 2000 \ldots 128000$ suggest that a power law gives a good fit. Regression analysis using a power law suggests that empty space grows as $n^{0.72}$. While this observation may be sufficient for a conjecture of sublinear growth, the resulting formula is greater than Shor's [13] asymptotic bound of $O((n \log n)^{2/3})$. Similar analysis at $u = .8$ suggest that the curve is growing faster than $n^{0.938}$; while this is evidence for asymptotic linearity, the result is not conclusive. At $u = .2$ empty space grows approximately as $n^{0.75}$. Empty space appears to approach its asymptote very slowly at low $u$, so it is hazardous to extend these results to predict asymptotic behavior.

A more useful approach is to study the structure of First Fit packings as $u$ varies. The analysis of Section 2 rests on showing that the rate at which bad bins (with gap in the range $(\alpha_0, \beta)$) are created is greater than the rate at which they are fixed. Bad bins form because there is a discontinuity in the candidate sequence of size approaching $\alpha_0 - .85$. As a general rule, the discontinuity appears between 2-bins and 1-bins: 1-bins cannot have total weight greater than .85, and most 2-bins tend to have total weight greater than $\alpha_0$. Instead of forming a smooth "filter" for new weights and promoting tight pairings of items with 1-bins, the discontinuity in candidate sequence allows bad pairings to occur.

*Conjecture: Two properties must hold in order to produce non-optimal First Fit packings: (1) a discontinuity appears with regularity in the candidate sequence, allowing bad bins to form; and (2) bad bins form rapidly enough that there aren't enough small items arriving to fix them.*

Recall that Bentley et al. [2] have shown that First Fit is optimal in the expected case when the weights
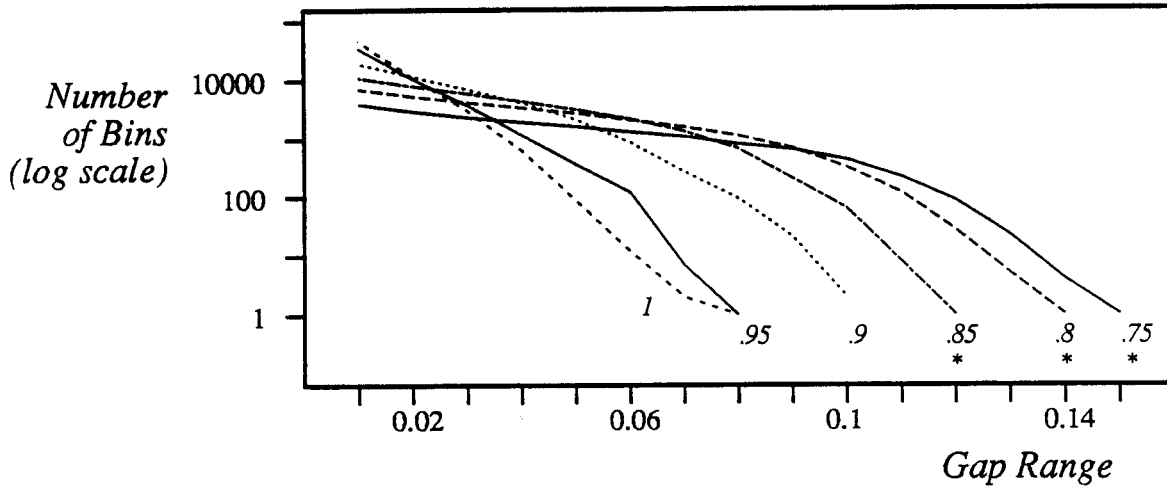
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Figure 4: Distribution of gaps in 2-bins for 6 values of $u$ ranging from .75 to 1. The y-axis is on a logarithmic scale.

are drawn from $(0, 1]$. It is easy to see that property (1) cannot hold for this input model. That is, there can be no discontinuity in the candidate sequence because the fullest 1-bin can have total weight 1, which is certainly as much as the emptiest 2-bin. Since the candidate sequence has no consistent discontinuities, the pairings are tight and the packing is optimal.

Is there a consistent discontinuity between 1-bins and 2-bins for $u$ near 1? Figure 4 depicts the distribution of gaps in 2-bins for $u = .75, .8, \ldots 1$. Each curve represents 1 trial at $n = 128000$ and the specified $u$ value. For each *gap range* $[g, g - .01)$ for $g = .01 \ldots .15$, a curve shows the number of 2-bins having gap in that range. Note that the y-axis is on a logarithmic scale. The three curves corresponding to $u = 1, .95, .9$ extend to the right of $1 - u$ (the fullest possible 1-bin). Therefore there is no apparent discontinuity between 2-bins and 1-bins for these sample points. On the other hand, the three starred

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curves, corresponding to $u = .85, .8, .75$, fall short of $1 - u$, indicating a discontinuity in the candidate sequence. This shortfall in 2-bins is observed consistently over repeated trials at these sample points.

Further (limited) simulations are unable to detect a gap in the candidate sequence when $u$ is greater than .86. It is quite possible, even likely, that our simulations are not able to display asymptotic behavior and that First Fit is non-optimal for any $u < 1$.

*Conjecture: There exists some $u_0 \in [.86, 1]$ such that First Fit gives optimal packings for $u > u_0$ and First Fit gives non-optimal packings for $.85 < u < u_0$.*

An interesting open problem is whether $u_0$ is strictly less than 1. Analysis of the number of 1-bins appearing for $u$ in this range might lead to bounds on $u_0$. Suppose, for example, that the number of 1-bins at any time in the packing is a slowly growing function of $n$ (it must be $o(n)$ or the proof of non-optimality would be immediate). The candidate sequence of 1-bins alone would grow unboundedly in $n$, producing a tight self-filter. This would promote very tight 2-bin pairings, tight enough so that a discontinuity in the packing sequence would appear. Therefore, by the first conjecture, the packing would be non-optimal.

Suppose on the other hand that the number of 1-bins in the packing is constant with respect to $n$. In this case, the tight self-filter on 1-bins would not exist, and gaps in 2-bins overlap those in 1-bins. The discontinuity disappears, producing an optimal packing. Note that this argument will not hold for all $u$: at $u=.85$ the number of 1-bins is constant and the packing is not optimal. In this case, even though the 1-bins are not filtered well, the 2-bins remain packed tightly enough so that there is no overlap with 1-bins. Further experiments to study the growth of 1-bins with respect to $n$ would be useful.
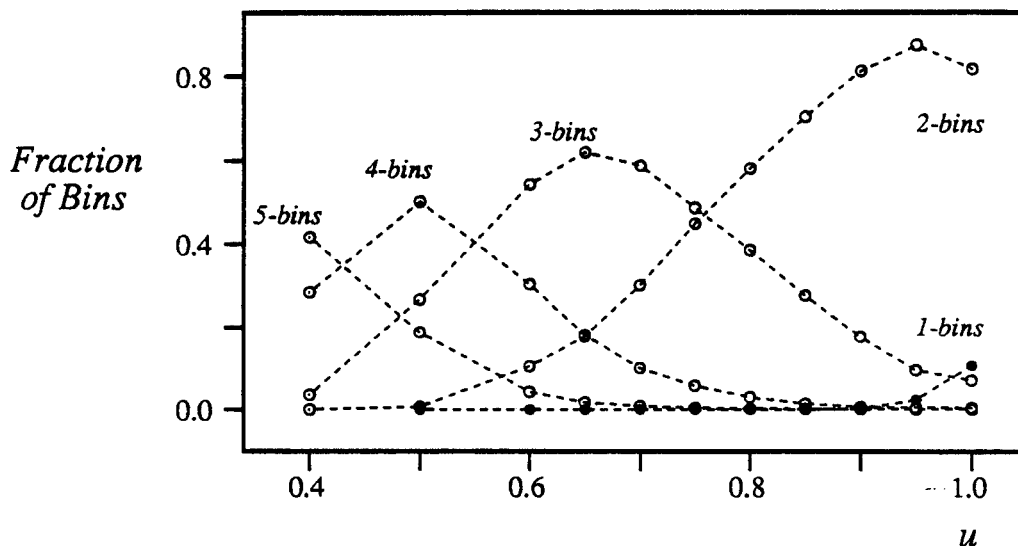
Figure 5: The fraction of k-bins in packings of weights with $u$ between .4 and 1.

Finally, consider the structure of First Fit packings for $u < .85$. Figure 5 shows the proportion of k-bins in the packings for $k = 1, 2 \ldots 5$ and for values of $u$ between .4 and 1. Each curve represents the mean of 5 trials at $n = 128000$. The 1-bin curve is darkened so that it may be seen more easily.

At $u = .85$ nearly 90% of the bins are 2-bins; this corresponds to our observation that 2-bins form more rapidly than small items arrive to fix them (forming 3-bins). As $u$ decreases away from .85, however, the proportion of 2-bins decreases and gives way to 3- and 4-bins.

At $u = 2/3$ there are certainly enough items to fix bad 2-bins, so property 2 of our main conjecture fails. The conjecture can be modified, however, to consider whether bad 3-bins are formed and whether there are enough small items to fix the bad 3-bins (forming 4-bins). Where previously we argued that the discontinuity between 2-bins and 1-bins causes items to pair badly with 1-bins, now we might argue that a discontinuity between 3-bins and 2-bins causes items to match badly with 2-bins. Limited experiments

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and preliminary exploration suggest, however, that there is no such discontinuity between 3-bins and 2-bins when $u \leq 2/3$.

*Conjecture: First Fit packings are optimal when $u \leq 2/3$.*

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