HYBRID AUTOMATA AND ε-ANALYSIS ON A NEURAL OSCILLATOR

A. Casagrande\textsuperscript{1} \quad T. Dreossi\textsuperscript{2} \quad C. Piazza\textsuperscript{2}

\textsuperscript{1}DMG, University of Trieste, Italy

\textsuperscript{2}DIMI, University of Udine, Italy
Intuitively...

Motivations:

- Reachability Analysis of Hybrid Automata with applications to Biological Models
- Automatic analysis (…as much as possible)
- Exploit existing “approximation” techniques
- Avoid ad-hoc studies

Case of study:

- Neural oscillator
- Rhythmic neural activity
- Interaction between neurons
Intuitively...

Saddle point

Avoid ad-hoc studies
Outline

1. Hybrid Automata
2. “Approximation” Techniques
3. Neural Oscillator Model
4. Computations
5. Conclusions
Hybrid Automata

Syntax

- **Finite state automaton** $H + \text{Continuous variables } Z, Z'$
- **State**: pair $\langle v, r \rangle$

Example: Thermostat
**Hybrid Automata**

**Semantics - Continuous Transition:** \( \langle \nu, r \rangle \xrightarrow{t} C \langle \nu, s \rangle \)

- **Continuous trajectory:** \( f_\nu : \mathbb{R}^{d(H)} \rightarrow (\mathbb{R}^+ \rightarrow \mathbb{R}^{d(H)}) \)
  where \( \text{Dyn}(\nu)[Z, Z', T] \) is \( Z' = f_\nu(Z)(T) \)
- \( r = f_\nu(r)(0), s = f_\nu(r)(t) \)
- **Invariant:** \( \text{Inv}(\nu)[f_\nu(r)(t')] \) for each \( t' \in [0, t] \)

![Diagram showing the transition between On and Off states with equations for \( Z_1 \) and \( Z'_1 \)]
Hybrid Automata

Semantics - Discrete Transition: \( \langle v, r \rangle \xrightarrow{e} D \langle v', s \rangle \)

- **Discrete jump**: \( e = \langle v, v' \rangle \)
- **Activation**: \( Act(e)[r] \)
- **Reset**: \( Res(e)[r, s] \)

\[
\begin{align*}
Z_1 &\leq 22 \\
Z_1' & = Z_1 e^T \\
\text{On} &
\end{align*}
\]

\[
\begin{align*}
Z_1 &\geq 18 \\
Z_1' & = Z_1 e^{-T} \\
\text{Off} &
\end{align*}
\]

\[
\begin{align*}
Z_1 &\geq 21 \\
Z_1' & = Z_1
\end{align*}
\]

\[
\begin{align*}
Z_1 &\leq 19 \\
Z_1' & = Z_1
\end{align*}
\]
Hybrid Automata Reachability $\Rightarrow$ Undecidable! [HKPV95]

“Approximation” Techniques:

- Noise and Disturbed Automata [Frä99]
- Approximate Bisimulations and Simulations [GP07]
- $\epsilon$-Semantics [CPP09]
- Template Polyhedral [SDI08]
- Abstractions for Hybrid Systems [Tiw08]
Techniques
Noise and Disturbed Automata [Frä99]

- Continuous variables provide unbounded quantity of memory
- Trajectories of real systems are subject to noise

**Definition**

$\tilde{H}$ is a disturbance variant of $H$ of noise level $\epsilon$ or more if for each pair of states $s, s'$ such that $\delta(s, s') < \epsilon$ it holds that if $r \xrightarrow{t} s$ in $H$, then $r \xrightarrow{t} s'$ in $\tilde{H}$.

- A hybrid automaton $H$ is robust if there exists an $\epsilon$ s.t.
if $a$ does not reach $a'$ in $H$, then $a$ does not reach $a'$ in $\tilde{H}$
- Noise ensures (semi-)decidability of reachability problem
Techniques

Approximate Bisimulations and Simulations [GP07]

- Relaxation of bisimulations and simulations
- Simplification of dynamics and resets
- Observation map: $\langle\langle \cdot \rangle\rangle : \mathbb{R}^{d(H)} \rightarrow \mathbb{R}^d$
Techniques

$\epsilon$-Semantics [CPP09]

- Infinite precision is an approximation of reality
- *Semantics* of a formula $\psi$ is the set of points $\{|\psi|\} \subseteq \mathbb{R}^n$ which satisfies $\psi$
- Formulae semantics of “dimension of at least $\epsilon$”
- $B(S, \epsilon) = \{ q \in \mathbb{R}^n \mid \exists p \in S(\delta(p, q) < \epsilon)\}$

**Example (Sphere Semantics)**

- $(|t_1 \circ t_2|)_\epsilon \overset{\text{def}}{=} B(|t_1 \circ t_2|, \epsilon)$, for $\circ \in \{=, <\}$
- $(|\psi_1 \land \psi_2|)_\epsilon \overset{\text{def}}{=} \bigcup_{B({p}, \epsilon) \subseteq (|\psi_1|)_\epsilon \cap (|\psi_2|)_\epsilon} B({p}, \epsilon)$
- $(\forall X \psi[X, Z]|)_\epsilon \overset{\text{def}}{=} (\bigwedge_{r \in \mathbb{R}} \psi[r, Z]|)_\epsilon$
Techniques

Comparison

$\epsilon$-bisimulation

$\epsilon$-semantics

Noise
Techniques

Computing $\epsilon$-Semantics

Translation for any $\varphi[X] \in T$ and $\epsilon \in \mathbb{R}_{>0}$

$\hat{\varphi}[X] \in T$ such that $(|\varphi[X]|)_{\epsilon} = \{|\hat{\varphi}[X]|\}$

**Example**

$((t_1 \circ t_2)[Y, W])_{\epsilon} \overset{\text{def}}{=} \exists W_0((t_1 \circ t_2)[Y, W_0] \land \delta(W_0, W) < \epsilon), \text{ for } \circ \in \{=, <\}$

$(\hat{\varphi}[Y, W] \land \hat{\psi}[Y, W])_{\epsilon} \overset{\text{def}}{=} \exists W_0(\forall W_1(\delta(W_0, W_1) < \epsilon \rightarrow ((\hat{\varphi})_{\epsilon} \land (\hat{\psi})_{\epsilon})[Y, W_1]) \land \delta(W_0, W) < \epsilon)$

$(\forall X \hat{\varphi}[Y, X, W])_{\epsilon} \overset{\text{def}}{=} \exists W_0(\forall W_1(\delta(W_0, W_1) < \epsilon \rightarrow \forall X_0((\hat{\varphi}[Y, X_0, W_1])_{\epsilon} \land \delta(W_0, W) < \epsilon))$
Neural Oscillator

Model

Continuous model [TMBD99]

\[ f(\tau, \lambda) : \begin{cases} 
\dot{X}_e &= -\frac{X_e}{\tau} + \tanh(\lambda \cdot X_e) - \tanh(\lambda \cdot X_i) \\
\dot{X}_i &= -\frac{X_i}{\tau} + \tanh(\lambda \cdot X_i) + \tanh(\lambda \cdot X_i) 
\end{cases} \]

Piecewise model

\[ \tilde{f}_\alpha(\tau, \lambda) : \begin{cases} 
\dot{X}_e &= -\frac{X_e}{\tau} + h_{\lambda, \alpha}(X_e) - h_{\lambda, \alpha}(X_i) \\
\dot{X}_i &= -\frac{X_i}{\tau} + h_{\lambda, \alpha}(X_e) + h_{\lambda, \alpha}(X_i) 
\end{cases} \]

\[ h_{\lambda, \alpha}(X) \text{ approximating } \tanh(\lambda \cdot X) \]
Hybrid Automaton $H\tilde{f}$ associated to $\tilde{f}_\alpha(\tau, \lambda)$
Neural Oscillator

Experimental results

\(\varepsilon\)-Noise:

- \(\langle 0, 0 \rangle\) is an unstable equilibrium
- \(H_{\tilde{f}}\) is fragile \(\Rightarrow\) \(\langle 0, 0 \rangle\) in \(\tilde{H}_{\tilde{f}}\) is different from \(\langle 0, 0 \rangle\) in \(H_{\tilde{f}}\)
- It is not possible to define robust models \(\Rightarrow\) Unsafe

\(\varepsilon\)-(Bi)Simulations:

- \(\varepsilon\)-simulation \(\Rightarrow\) Reduction of analysis complexity
- \(\langle 0, 0 \rangle\) is a singularity of the model:
  - \(\langle 0, 0 \rangle\) neighborhood reaches the limit cycle
  - No \(\varepsilon\)-(bi)simulation relates \(\langle 0, 0 \rangle\) with any other state
**Neural Oscillator**

**Experimental results**

\(\epsilon\)-Semantics:

1. Approximating (polynomials) solutions of differential equations
2. Translating \(\epsilon\)-semantics (Sphere Semantics)
3. Computing exploiting cylindrical algebraic decomposition

Original evolution  
First degree approximation evolution
Neural Oscillator

Experimental results

$\epsilon$-Semantics (formally and automatically proves that):

- Flow tube including limit cycle
- $\langle 0, 0 \rangle$ reaches the limit flow tube
- No need to modify the automaton (syntax)

but . . .

- $\epsilon$-semantics translation increases fromulæ complexity
- Direct computation of reach set $\Rightarrow$ high complexity
- Reformulation of the problem in form of a property:

"Convergence to the limit cycle"
Convergence

and Formulae reduction
Conclusions:

- $\epsilon$-semantics better reflects the real system behaviour
- ... high computational complexity

Future works:

- Combine several hybrid automata
- Integrations of numerical approximation techniques
- Automatic simplification of the formulæ
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