CONJUGATE POINTS IN FORMATION CONSTRAINED OPTIMAL MULTI-AGENT COORDINATION: A CASE STUDY*

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Abstract. In this paper, an optimal coordinated motion planning problem for multiple agents subject to constraints on the admissible formation patterns is formulated. Solutions to the problem are reinterpreted as distance minimizing geodesics on a certain manifold with boundary. A geodesic on this manifold may fail to be a solution for different reasons. In particular, if a geodesic possesses conjugate points, then it will no longer be distance minimizing beyond its first conjugate point. We study a particular instance of the formation constrained optimal coordinated motion problem, where a number of initially aligned agents tries to switch positions by rotating around their common centroid. The complete set of conjugate points of a geodesic naturally associated to this problem is characterized analytically. This allows us to prove that the geodesic will not correspond to an optimal coordinated motion when the angle of rotation exceeds a threshold that decreases to zero as the number of agents increases. Moreover, infinitesimal perturbations that improve the performance of the geodesic after it passes the conjugate points are also determined, which interestingly are characterized by a certain family of orthogonal polynomials.

Key words. Conjugate point, multi-agent coordination, geodesics, orthogonal polynomials.

AMS subject classifications. 53C22, 58E25, 05E35.

1. Introduction. Multi-agent coordinated motion planning problems arise in various contexts, such as Air Traffic Management (ATM) [11, 21], robotics [3], and Unmanned Aerial Vehicle (UAVs [17, 20, 22]). In most cases, certain separation constraints between the agents have to be guaranteed due to physical, safety, or efficiency reasons. For example, in ATM systems, aircraft flying at the same altitude are required to maintain a minimal horizontal separation of 5 nautical miles in en-route airspace and 3 nautical miles close to airports. When multiple mobile robots are performing a coordinated task such as lifting a common object, specific formations have to be kept by the robots throughout the operation. For UAVs, flying in formation may reduce the fuel expenditure and the communication power needed for information exchange.

In this paper, we formulate an optimal coordinate motion planning problem for multiple agents under formation constraints. We consider all the coordinated motions that can lead a group of agents from given initial positions to given destination positions within a certain time horizon, while satisfying the additional constraint that the formation patterns of the agents belong to a prescribed subset. Among the coordinated motions in this restricted set, we try to find the ones that minimize a weighted sum of the energy functions of the individual agents' motions, with the weights representing agent priorities. In our problem formulation, we use simple kinetic models for the agent dynamics, and consider only holonomic constraints, as opposed to the numerous work dealing with nonholonomic constraints (e.g. [2, 10, 19]). See, for example, [5, 12, 15], for relevant work on the problems of stable and optimal coordinated

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control of vehicle formations.

A geometric interpretation of the considered optimal coordinated motion planning problem is given in this paper. According to this interpretation, a solution to the problem is a shortest curve with constant speed between two fixed points in a certain manifold with boundaries, with boundaries determined by the feasible formation patterns. Being a shortest curve between two points that can be far away from each other, such a *globally* distance-minimizing curve is obviously also a *locally* distance-minimizing curve, i.e., a curve whose sufficiently short segments are distance-minimizing between their respective end points. Locally distance-minimizing curves parameterized with constant speed are called *geodesics*. Thus a solution to the problem necessarily corresponds to a geodesic of the manifold.

Conversely, however, for various reasons a geodesic of the manifold may fail to be globally distance-minimizing, thus failing to solve the problem. One of the reasons is the occurrence of conjugate points. Traveling along a geodesic from a fixed starting point, a conjugate point occurs at a point where there exists a non-trivial Jacobi field along the geodesic vanishing at both the starting point and that particular point [4], or less rigorously, where there exists infinitesimally more than one geodesic connecting the starting point to that point. For a simple example, consider the sphere. Geodesics on the sphere are great circles; and conjugate points along a great circle occur at the anti-podal point of its starting point. It is a well known fact in Riemannian geometry that a geodesic will not be distance minimizing once it passes its first conjugate point [4], as one can then infinitesimally perturb it to obtain a shorter curve with the same end points. In the sphere example, when a great circle extends beyond two anti-podal points, a shorter curve between its starting and ending points can be found by following the intersection of the sphere with any co-dimensional one plane passing through the two points. The aim of this paper is to study through a concrete example the loss of optimality due to the existence of conjugate points in multi-agent coordination problem, which so far has been largely ignored in the literature.

It is in general difficult, if not impossible, to characterize the conjugate points of a geodesic analytically. In this paper, we shall focus on a special instance of the formation constrained optimal coordination problem, namely, a group of initially aligned agents switching positions by rotating around their common centroid. We shall show that the conjugate points of a geodesic that arises naturally when solving this particular problem admit nice analytic formulae. We shall also determine the infinitesimal perturbations that can shorten the geodesic with various efficiencies once it passes its conjugate points, and characterize them using a certain family of orthogonal polynomials. Our results can be interpreted geometrically as characterizing how long one can travel along the outer edge of a (high-dimensional) "donut" before the resulting curve is no longer distance-minimizing, or mechanically, how a multi-segment snake-like robot should turn on the ground optimally starting from the configuration in which it is completely stretched, with all its segments aligned in a straight line.

As a preview of the results and a specific example, see Fig. 1.1, where five helicopters initially flying in a straight line try to reach new positions by rotating counterclockwise around their centroid, i.e., the middle helicopter, at the same angular velocity during the time period $[0, 1]$ (thus the five helicopters form a straight line at all times). The results in this paper will show that this joint maneuver is optimal in terms of minimizing a cost function defined as the sum of energy of individual helicopter's manuevers only if the angle τ of rotation is small, and that it is not optimal if $\tau > \frac{\pi}{3}$. Indeed, if for example $\tau = \pi$, then we can find better manuevers than the one

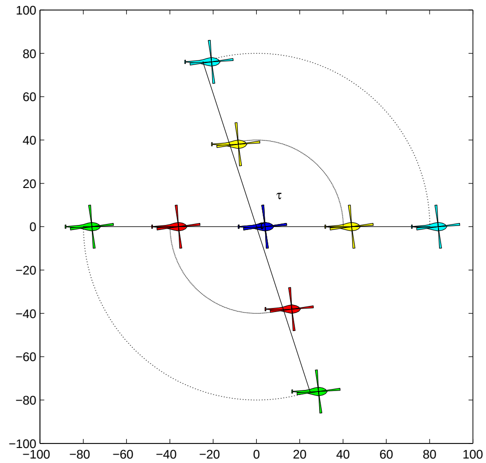

FIG. 1.1. *A five-agent example*

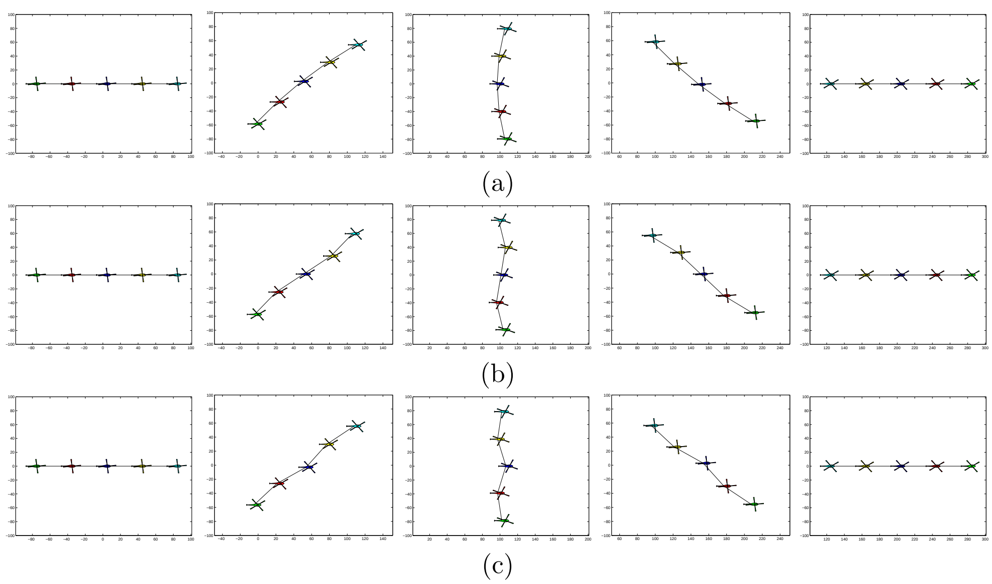

FIG. 1.2. *Three joint maneuvers with increasingly lower cost than the one shown in Fig. 1.1 when $\tau = \pi$. Top row: maneuver (a); middle row: maneuver (b); bottom row: maneuver (c). From left to right: snapshots at time $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, respectively.*

in Fig. 1.1. We plot in the three rows of Fig. 1.2 three such maneuvers, where in each row the five figures from left to right represent the snapshots of the maneuver at time $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, respectively. In terms of performance, we will show that maneuver (c) is the best; maneuver (a) is the worst of the three, but still better than the original one in Fig. 1.1.

The paper is organized as follows. In Section 2 the formation patterns of a group of agents are defined and the problem of formation constrained optimal multi-agent coordination is formulated. In Section 3 the conjugate points of the geodesic naturally arising in a particular instance of the problem are characterized analytically. Infinitesimal perturbations that improve the performance of the geodesic beyond its conjugate points are also determined. Finally, some conclusions are drawn in Section 4.

2. Formation Constrained Optimal Multi-Agent Coordination. In this section we formulate the problem of formation constrained optimal multi-agent coordination, and describe some properties of its solutions that will be useful in the next section. For simplicity, the problem is formulated in Euclidean spaces. See [8] for an extension of the results to general Riemannian manifolds with a group of symmetries.

2.1. Problem Formulation. Let $\mathbb{R}^n$ be the Euclidean space with the standard Euclidean metric. Denote by $\langle q_i \rangle_{i=1}^k = (q_1, \ldots, q_k)$ an (ordered) k-tuple of points in $\mathbb{R}^n$ for some positive integer k. We say that $\langle q_i \rangle_{i=1}^k$ satisfies the *r-separation condition* for some $r > 0$ if $d(q_i, q_j) = \|q_i - q_j\| \geq r$ for all $i \neq j$, i.e., if the minimum pairwise distance among the k points is at least r.

Consider k agents moving in $\mathbb{R}^n$. Suppose that they start at time 0 from the initial positions $a_1, \ldots, a_k \in \mathbb{R}^n$ and must reach the destination positions $b_1, \ldots, b_k \in \mathbb{R}^n$ at time t_f. We assume that both $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$ satisfy the r-separation condition. Denote the joint trajectory of the agents by a k-tuple of curves $\gamma = \langle \gamma_i \rangle_{i=1}^k$, where the trajectory of agent i during the time interval $[0, t_f]$ is modeled as a continuous and piecewise C^1 curve $\gamma_i : [0, t_f] \to \mathbb{R}^n$ such that $\gamma_i(0) = a_i$ and $\gamma_i(t_f) = b_i$. γ is said to be *collision-free* if for all $t \in [0, t_f]$ the k-tuple $\langle \gamma_i(t) \rangle_{i=1}^k$ satisfies the r-separation condition, or equivalently, if the distance between any two agents is at least r at all times during $[0, t_f]$. Define the cost of the joint trajectory γ as

$$J(\gamma) = \sum_{i=1}^{k} \mu_i E(\gamma_i), \tag{2.1}$$

where $\mu_1, \ldots, \mu_k$ are positive numbers representing the priorities of the agents, and

$$E(\gamma_i) = \frac{1}{2} \int_0^{t_f} \|\dot{\gamma}_i(t)\|^2 \, dt \tag{2.2}$$

is the standard energy of the trajectory γ_i as a curve in $\mathbb{R}^n$, for $i = 1, \ldots, k$. If we denote by $L(\gamma_i) = \int_0^{t_f} \|\dot{\gamma}_i(t)\| \, dt$ the arc length of γ_i, then by the Cauchy-Schwartz inequality [13], $E(\gamma_i) \geq \frac{1}{2} L^2(\gamma_i)/t_f$, where the equality holds if and only if $\|\dot{\gamma}_i(t)\|$ is constant for $t \in [0, t_f]$.

Based on the introduced notations and concepts, the problem of optimal coordinated motions under the r-separation constraint can be formulated as follows.

Problem 1 (Optimal Collision Avoidance (OCA)) *Among all the collision-free joint trajectories $\gamma = \langle \gamma_i \rangle_{i=1}^k$ that start from $\langle a_i \rangle_{i=1}^k$ at time 0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_f, find the ones that minimize the cost $J(\gamma)$.*

To each k-tuple $\langle q_i \rangle_{i=1}^k$ satisfying the r-separation constraint we associate an undirected graph $(\mathcal{V}, \mathcal{E})$ with set of vertices $\mathcal{V} = \{1, \ldots, k\}$ and set of edges $\mathcal{E} = \{(i,j) : \|q_i - q_j\| = r\}$. We call $(\mathcal{V}, \mathcal{E})$ the *formation pattern* of the k-tuple $\langle q_i \rangle_{i=1}^k$.

Remark 1 *For given n, r and k, not all graphs with k vertices can represent the formation pattern of some k-tuple of points of $\mathbb{R}^n$ satisfying the r-separation condition. For example, if $n = 2$ and $k = 4$, the complete graph with four vertices and edges between each pair of them is not the formation pattern of any $\langle q_i \rangle_{i=1}^4$ satisfying the r-separation condition, regardless of r.*

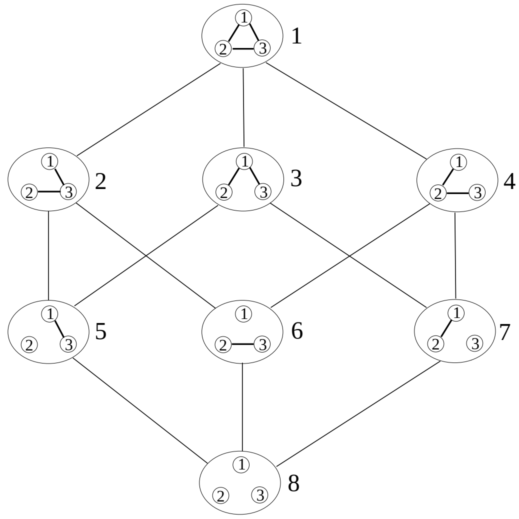

FIG. 2.1. *Hasse diagram of $\mathcal{F}$ in the case $n = 2$ and $k = 3$.*

Denote by $\mathcal{F}$ the set of *feasible* formation patterns, i.e., the set of all undirected graphs $(\mathcal{V}, \mathcal{E})$ associated with k-tuples of points satisfying the r-separation condition. A partial order $\prec$ is defined on $\mathcal{F}$ such that $(\mathcal{V}, \mathcal{E}_1) \prec (\mathcal{V}, \mathcal{E}_2)$ for $(\mathcal{V}, \mathcal{E}_1)$ and $(\mathcal{V}, \mathcal{E}_2)$ in $\mathcal{F}$ if and only if $(\mathcal{V}, \mathcal{E}_1)$ is a subgraph of $(\mathcal{V}, \mathcal{E}_2)$. Based on this partial order relation, $\mathcal{F}$ can be rendered graphically as a *Hasse diagram* [14]. In this diagram, each element of $\mathcal{F}$ is represented by a node on a plane at a certain position such that the node corresponding to $(\mathcal{V}, \mathcal{E}_1)$ is placed below the node corresponding to $(\mathcal{V}, \mathcal{E}_2)$ if $(\mathcal{V}, \mathcal{E}_1) \prec (\mathcal{V}, \mathcal{E}_2)$, and a line segment is drawn upward from node $(\mathcal{V}, \mathcal{E}_1)$ to node $(\mathcal{V}, \mathcal{E}_2)$ if and only if $(\mathcal{V}, \mathcal{E}_1) \prec (\mathcal{V}, \mathcal{E}_2)$ and there exists no other $(\mathcal{V}, \mathcal{E}) \in \mathcal{F}$ such that $(\mathcal{V}, \mathcal{E}_1) \prec (\mathcal{V}, \mathcal{E})$ and $(\mathcal{V}, \mathcal{E}) \prec (\mathcal{V}, \mathcal{E}_2)$. As an example, Fig. 2.1 plots the Hasse diagram of $\mathcal{F}$ in the case $n = 2$ and $k = 3$.

The formation pattern of a joint trajectory γ at time $t \in [0, t_f]$ is the formation pattern of the k-tuple $\langle \gamma_i(t) \rangle_{i=1}^k$, and is time-varying. In certain applications the following problem arises, which is a restrictive version of the OCA problem.

Problem 2 (Optimal Formation-Constrained Coordination (OFC)) *Among all the collision-free joint trajectories $\gamma = \langle \gamma_i \rangle_{i=1}^k$ that start from $\langle a_i \rangle_{i=1}^k$ at time 0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_f, find the ones that minimize the cost $J(\gamma)$ and satisfy the constraint that the formation pattern of γ at any time $t \in [0, t_f]$ belongs to some prescribed subset $\tilde{\mathcal{F}}$ of $\mathcal{F}$. Here we assume that $\tilde{\mathcal{F}}$ contains the formation patterns of both $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$.*

In the case $\tilde{\mathcal{F}} = \mathcal{F}$ the OFC problem reduces to the OCA problem.

2.2. Geometric Interpretation. The OCA and the OFC problems can be interpreted in the following geometric way.

Each k-tuple $\langle q_i \rangle_{i=1}^k$ of points in $\mathbb{R}^n$ corresponds to a single point $q = (q_1, \ldots, q_k)$ in $\mathbb{R}^{nk} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k times). Thus each joint trajectory $\gamma = \langle \gamma_i \rangle_{i=1}^k$ of the k agents corresponds to a curve γ in $\mathbb{R}^{nk}$ starting from $a = (a_1, \ldots, a_k)$ at time 0 and ending at $b = (b_1, \ldots, b_k)$ at time t_f. The collision-free condition is equivalent to the curve γ avoiding the obstacle

$$W \triangleq \cup_{i \neq j} \{(q_1, \dots, q_k) \in \mathbb{R}^{nk} : \|q_i - q_j\| < r\}. \tag{2.3}$$

If $\mu_1 = \cdots = \mu_k = 1$, the cost $J(\gamma)$ in (2.1) is the standard energy of γ as a curve in $\mathbb{R}^{nk}$. For general $\langle \mu_i \rangle_{i=1}^k$, $J(\gamma)$ is the standard energy of γ as a curve in $\mathbb{R}^{nk}$ after appropriately scaling the coordinate axes of $\mathbb{R}^{nk}$.

Consider $\mathbb{R}^{nk} \setminus W$, a manifold with boundary whose metric is inherited from the standard one on $\mathbb{R}^{nk}$ after properly scaling its coordinate axes. The above discussion suggests that solutions to the OCA problem are energy-minimizing curves connecting a and b in $\mathbb{R}^{nk} \setminus W$. It is well known in Riemannian geometry [16] (see also the discussion after equation (2.2)) that such curves are necessarily the shortest curves in $\mathbb{R}^{nk} \setminus W$ parameterized with constant speed. Therefore, solving the OCA problem is equivalent to finding the geodesics from a to b in $\mathbb{R}^{nk} \setminus W$ that are also globally distance-minimizing.

As for the OFC problem, regarded as a curve in $\mathbb{R}^{nk} \setminus W$, a solution to the OFC problem can only lie in a *subset* of $\mathbb{R}^{nk} \setminus W$ obtained by piecing together cells of various dimensions, one for each admissible formation pattern in $\tilde{\mathcal{F}}$. Depending on $\tilde{\mathcal{F}}$, this union of cells can be highly complicated. As an example, in Fig. 2.1 one can choose $\tilde{\mathcal{F}}$ to consist of formation patterns 1, 2, 3, and 4, thus requiring that every two agents "contact" each other either directly or indirectly via the third agent at all time instants. This makes sense in practical situations where the three agents have to share some common data and information exchange is possible only at the minimum allowed distance. As another example, $\tilde{\mathcal{F}}$ can be chosen to consist of formation patterns 1, 3, 4, and 7. In this case agent 1 and agent 2 are required to be bound together during the whole time interval $[0, t_f]$, and the OFC problem can be viewed as the OCA problem between agent 3 and this two-agent subsystem. Solutions to the OFC problem are the distance-minimizing geodesics connecting a to b in this union of cells after proper scaling of coordinate axes.

Remark 2 *Solutions to the OCA and OFC problems may not exist. The OCA problem of two agents on a line trying to switch positions is one such example. The geometric interpretation of the OCA and OFC problems allows us to easily formulate conditions for their feasibility. In general, to ensure the existence of solutions, it is sufficient (though not necessary) to require that the subset of $\mathbb{R}^{nk} \setminus W$ corresponding to $\tilde{\mathcal{F}}$ is closed and that a and b are in the same connected component of this subset. The first requirement translates into the following property of $\tilde{\mathcal{F}}$: for each $(\mathcal{V}, \mathcal{E}) \in \tilde{\mathcal{F}}$, any formation pattern $(\mathcal{V}, \mathcal{E}_1)$ such that $(\mathcal{V}, \mathcal{E}) \prec (\mathcal{V}, \mathcal{E}_1)$ is also an element of $\tilde{\mathcal{F}}$. The second requirement is satisfied if there exists at least one collision-free joint trajectory $\langle \gamma_i \rangle_{i=1}^k$ from a to b whose formation pattern is in $\tilde{\mathcal{F}}$ at any time $t \in [0, t_f]$.*

2.3. Conservation Law for the Solutions. We now describe some properties of the solutions to the OFC problem that will be used in the next sections.

Proposition 1 *Suppose that the joint trajectory $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is a solution to the OFC problem. Then the quantities*

$$\sum_{i=1}^{k} \mu_i \dot{\gamma}_i(t), \quad \sum_{i=1}^{k} \mu_i \big(\gamma_i(t) \dot{\gamma}_i^T(t) - \dot{\gamma}_i(t) \gamma_i^T(t)\big) \tag{2.4}$$

are constant for all $t \in [0, t_f]$.

Note that in this paper an element in $\mathbb{R}^n$ is regarded by default as a column vector; thus the second quantity in (2.4) is an n-by-n matrix.

If one thinks of each agent i as a point in $\mathbb{R}^n$ with mass μ_i, then Proposition 1 implies that the linear and (generalized) angular momenta of the k-point mass system are conserved along the solutions to the OFC problem. The proof of this proposition can be found in, e.g., [6], and, in the case of general Riemannian manifolds with a group of symmetries, in [8]. Thus we omit the proof here.

One implication of Proposition 1 is that, if both a and b are μ-aligned in the sense that $\sum_{i=1}^k \mu_i a_i = \sum_{i=1}^k \mu_i b_i = 0$, then so is any k-tuple $\langle \gamma_i(t)\rangle_{i=1}^k$, $t \in [0, t_f]$, for a solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OFC problem, namely,

$$\sum_{i=1}^k \mu_i \gamma_i \equiv 0.$$

Hence in the μ-aligned case the solution γ as a curve in $\mathbb{R}^{nk}$ must lie in a subspace

$$V = \{(q_1, \dots, q_k) \in \mathbb{R}^{nk} : \sum_{i=1}^k \mu_i q_i = 0\}$$

of $\mathbb{R}^{nk}$. This reduces the dimension of the state space by n.

3. An Interesting Example. Consider the OFC problem on $\mathbb{R}^2$ with the k agents having the same priority $\mu_1 = \cdots = \mu_k = 1$ and the minimal allowed separation $r = 1$. Suppose that the starting positions $\langle a_i \rangle_{i=1}^k$ of the agents are given by $\langle (\frac{2i-k-1}{2}, 0)\rangle_{i=1}^k$. In other words, at time $t = 0$, the k agents are aligned on the x-axis with common centroid at the origin and with consecutive agents at the minimal allowed separation. For each $t \geq 0$, denote by $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation of $\mathbb{R}^2$ by an angle t in radians. Suppose that the destination positions are $\langle b_i \rangle_{i=1}^k = \langle R_{t_f}(a_i)\rangle_{i=1}^k$. Both the initial and destination positions have the same formation pattern $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, k\}$ and $\mathcal{E} = \{(i, i+1) : i = 1, \dots, k-1\}$. We choose the admissible formation pattern set $\tilde{\mathcal{F}}$ to consist of this formation pattern only. Therefore, in considering the OFC problem, we require that agents i and $i+1$ are kept at constant distance r throughout $[0, t_f]$ for $i = 1, \dots, k-1$, and all other pairs of agents maintain a distance greater than r.

Since $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$ are μ-aligned, by the discussion after Proposition 1 in Section 2.3, a solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the above OFC problem satisfies $\sum_{i=1}^k \gamma_i \equiv 0$. Thus γ as a curve in $\mathbb{R}^{2k} \setminus W$ lies in the subspace $V = \{(q_1, \dots, q_k) \in \mathbb{R}^{2k} : \sum_{i=1}^k q_i = 0\}$. Furthermore, due to the admissible formation pattern set $\tilde{\mathcal{F}}$, γ belongs to the subset N of $(\mathbb{R}^{2k} \setminus W) \cap V$ given by

$$N = \{(q_1, \dots, q_k) \in \mathbb{R}^{2k} : \|q_i - q_{i+1}\| = 1, i = 1, 2, \dots, k-1,$$
$$\|q_i - q_j\| > 1, \forall j > i+1, \text{ and } \sum_{i=1}^k q_i = 0\}.$$

Therefore, according to Section 2.2, the solutions to the OFC problem are the distance-minimizing geodesics in N connecting $a = (a_1, \dots, a_k)$ to $b = (b_1, \dots, b_k)$.

In this section, we shall consider $\gamma^* = \langle R_t(a_i)\rangle_{i=1}^k$, $t \in [0, t_f]$, as a natural candidate solution to the above OFC problem[1]. Under the motions specified by γ^*, the

[1] In the following γ^* is also sometimes used to denote the whole curve $\langle R_t(a_i)\rangle_{i=1}^k$, $t \geq 0$. In such cases, its meaning should be clear from the context.

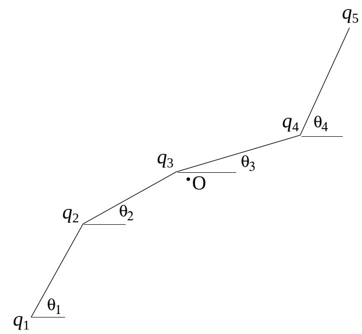

FIG. 3.1. *Coordinates of the manifold N when $k = 5$.*

k agents rotate counterclockwise at constant unit angular velocity around the origin from their starting to their destination positions. We shall show that $\langle R_t(a_i)\rangle_{i=1}^k$, $t \geq 0$, is a geodesic in N, thus γ^* is optimal for t_f small enough. We shall also show that the first conjugate point along this geodesic occurs at $\langle R_{\tau_k}(a_i)\rangle_{i=1}^k$ for some time τ_k, implying that γ^* is no longer optimal if $t_f > \tau_k$. We shall derive the analytical expression of τ_k, and show that $\tau_k \sim \frac{1}{k} \to 0$ as $k \to \infty$.

3.1. Geometry of the Manifold N. We start by constructing a convenient coordinate system on N, and then proceed to derive the geometry of N as a submanifold of $\mathbb{R}^{2k}$ in this coordinate system, such as its Riemannian metric, its covariant derivatives, and its curvature tensors. A general reference on Riemannian geometry can be found in [4].

First of all, N is a $(k-1)$-dimensional smooth submanifold of $\mathbb{R}^{2k}$, and admits global coordinates $(\theta_1, \ldots, \theta_{k-1})$, where θ_i is the angle $q_{i+1} - q_i \in \mathbb{R}^2$ makes with respect to the positive x-axis (see Fig. 3.1). The coordinate map $f : (\theta_1, \ldots, \theta_{k-1}) \mapsto (q_1, \ldots, q_k) \in N$ is defined by

$$q_i = q_1 + \sum_{j=1}^{i-1} \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}, \quad i = 2, \ldots, k, \tag{3.1}$$

where q_1 is chosen such that $\sum_{i=1}^k q_i = 0$, namely,

$$q_1 = -\frac{1}{k} \sum_{j=1}^{k-1} (k-j) \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}. \tag{3.2}$$

In these coordinates, γ^* corresponds to $\theta_i(t) = t$, $t \in [0, t_f]$, $i = 1, \ldots, k-1$.

At any $q \in N$, $\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_{k-1}}$ form a basis of T_qN. In this basis, the Riemannian metric $\langle\cdot,\cdot\rangle$ that N inherits from $\mathbb{R}^{2k}$ as a submanifold can be computed as

$$g_{ij} \triangleq \langle \frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j} \rangle = \langle \frac{\partial f}{\partial\theta_i}, \frac{\partial f}{\partial\theta_j} \rangle_{\mathbb{R}^{2k}}, \quad 1 \leq i, j \leq k-1. \tag{3.3}$$

Here f is the map defined in (3.1) and (3.2), and each $\frac{\partial f}{\partial\theta_i}$ is a vector in $\mathbb{R}^{2k}$. $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2k}}$ is the standard inner product on $\mathbb{R}^{2k}$. With this definition of metric, the map f becomes an isometry, and the cost of a joint trajectory γ of the k-agent system given by (2.1) can be expressed in two equivalent ways: in $(q_1, \ldots, q_k)$ coordinates it is

$$J(\gamma) = \frac{1}{2}\int_0^{t_f} \sum_{i=1}^{k} \|\dot{q}_i\|^2 \, dt,$$

and in $(\theta_1, \ldots, \theta_k)$ coordinates it is

$$J(\gamma) = \frac{1}{2}\int_0^{t_f} \left\langle \sum_{i=1}^{k-1} \dot{\theta}_i \frac{\partial}{\partial\theta_i}, \sum_{i=1}^{k-1} \dot{\theta}_i \frac{\partial}{\partial\theta_i} \right\rangle dt = \frac{1}{2}\int_0^{t_f} \sum_{i,j=1}^{k-1} g_{ij}\dot{\theta}_i\dot{\theta}_j \, dt.$$

After some careful computation, (3.3) in our case yields

$$g_{ij} = \Delta_{ij}\cos(\theta_i - \theta_j),$$

where Δ_{ij} are constants given by

$$\Delta_{ij} = \begin{cases} \frac{i(k-j)}{k} & \text{if } i \le j, \\ \frac{(k-i)j}{k} & \text{if } i > j. \end{cases} \tag{3.4}$$

The following lemma can be verified directly.

Lemma 1 *Let $\Delta = (\Delta_{ij})_{1\le i,j\le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ be the symmetric matrix with components Δ_{ij} defined in (3.4). Then*

$$\Delta^{-1} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Denote by $(g^{ij})_{1\le i,j\le k-1}$ the inverse matrix of $(g_{ij})_{1\le i,j\le k-1}$. The covariant derivative with respect to the Levi-Civita connection on N is given by [4]

$$\nabla_{\frac{\partial}{\partial\theta_i}} \frac{\partial}{\partial\theta_j} = \sum_{m=1}^{k-1} \Gamma_{ij}^m \frac{\partial}{\partial\theta_m}, \tag{3.5}$$

where Γ_{ij}^m are the Christoffel symbols defined as

$$\Gamma_{ij}^m = \frac{1}{2}\sum_{l=1}^{k-1}\left\{\frac{\partial g_{jl}}{\partial\theta_i} + \frac{\partial g_{li}}{\partial\theta_j} - \frac{\partial g_{ij}}{\partial\theta_k}\right\} g^{lm}, \quad 1 \le i,j,m \le k-1.$$

A curve γ in N is a geodesic if and only if $\nabla_{\dot{\gamma}}\dot{\gamma} \equiv 0$. By definition (3.5), this equation, also called the geodesic equation, can be written in the $(\theta_1, \ldots, \theta_{k-1})$ coordinates as a group of second order differential equations:

$$\ddot{\theta}_m = \sum_{i,j=1}^{k-1} \Gamma_{ij}^m \dot{\theta}_i\dot{\theta}_j, \quad m = 1, \ldots, k-1. \tag{3.6}$$

In our case, we can compute that, for $1 \le i,j,m \le k-1$,

$$\Gamma_{ij}^m = \begin{cases} 0 & \text{if } i \ne j, \\ \sum_{l=1}^{k-1} \Delta_{il}\sin(\theta_l - \theta_i) g^{lm} & \text{if } i = j. \end{cases} \tag{3.7}$$

Notice that along γ^* we have $\theta_1 = \cdots = \theta_{k-1}$. Therefore,

Lemma 2 *Along γ^* we have $\Gamma_{ij}^m = 0$ for all i, j, m, hence $\nabla_{\frac{\partial}{\partial\theta_i}} \frac{\partial}{\partial\theta_j} = 0$ for all i, j.*

Since $\dot{\gamma}^* = \frac{\partial}{\partial\theta_1} + \cdots + \frac{\partial}{\partial\theta_{k-1}}$, by Lemma 2 and the linearity of covariant derivatives,

$$\nabla_{\dot{\gamma}^*} \frac{\partial}{\partial\theta_j} = \nabla_{\sum_{i=1}^{k-1} \frac{\partial}{\partial\theta_i}} \frac{\partial}{\partial\theta_j} = \sum_{i=1}^{k-1} \nabla_{\frac{\partial}{\partial\theta_i}} \frac{\partial}{\partial\theta_j} = 0, \quad j = 1, \ldots, k-1. \tag{3.8}$$

Thus,

$$\nabla_{\dot{\gamma}^*} \dot{\gamma}^* = \nabla_{\dot{\gamma}^*} \sum_{j=1}^{k-1} \frac{\partial}{\partial\theta_j} = \sum_{j=1}^{k-1} \nabla_{\dot{\gamma}^*} \frac{\partial}{\partial\theta_j} = 0,$$

which is exactly the condition for γ^* to be a geodesic in N. Alternatively, since $\Gamma_{ij}^m = 0$ along γ^*, the geodesic equation (3.6) reduces to $\ddot{\theta}_m = 0$, $m = 1, \ldots, k-1$, which are trivially satisfied by $\gamma^*(t)$ with coordinates $\theta_i(t) = t$, $i = 1, \ldots, k-1$, $t \in [0, t_f]$.

Corollary 1 *γ^* is a geodesic in N.*

Since by definition a geodesic is locally distance-minimizing, we conclude that γ^* is a solution to the OFC problem for t_f small enough.

3.2. Conjugate Points along γ^*. To characterize the conjugate points of γ^*, we need to calculate the curvature of N. The curvature tensor of N is given by [4]

$$R(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}) \frac{\partial}{\partial\theta_l} = \sum_{m=1}^{k-1} R_{ijl}^m \frac{\partial}{\partial\theta_m}, \tag{3.9}$$

where R_{ijl}^m are defined from the Christoffel symbols as

$$R_{ijl}^m = \sum_{\beta=1}^{k-1} \Gamma_{il}^\beta \Gamma_{j\beta}^m - \sum_{\beta=1}^{k-1} \Gamma_{jl}^\beta \Gamma_{i\beta}^m + \frac{\partial \Gamma_{il}^m}{\partial\theta_j} - \frac{\partial \Gamma_{jl}^m}{\partial\theta_i}, \quad 1 \le i, j, l, m \le k-1. \tag{3.10}$$

A Jacobi field X along γ^* is a vector field along γ^* satisfying the Jacobi equation

$$\nabla_{\dot{\gamma}^*} \nabla_{\dot{\gamma}^*} X + R(\dot{\gamma}^*, X) \dot{\gamma}^* = 0. \tag{3.11}$$

A conjugate point $\gamma^*(\tau)$ along γ^* occurs at time $t = \tau$ if there is a non-trivial Jacobi field X along γ^* that vanishes at both time 0 and τ, i.e., $X(0) = X(\tau) = 0$. In the following, we shall characterize the conjugate points of γ^* starting from $\gamma^*(0) = a$.

Write an arbitrary vector field X along γ^* in coordinates as

$$X = \sum_{i=1}^{k-1} x_i \frac{\partial}{\partial\theta_i} \tag{3.12}$$

for some C^2 functions $x_i : [0, t_f] \to \mathbb{R}$. Note that here for simplicity we drop the time t in the above expression, following the convention in [4]. The explicit form for the vector field X should be $X(t) = \sum_{i=1}^{k-1} x_i(t) \frac{\partial}{\partial\theta_i}\big|_{\gamma^*(t)}$, $t \in [0, t_f]$. Equation (3.12) thus represents X as a vector $x = (x_1, \ldots, x_{k-1})$ in $\mathbb{R}^{k-1}$ that varies with time $t \in [0, t_f]$.

By equation (3.8) and the property of covariant derivatives, we have

$$\nabla_{\dot\gamma^*} X = \nabla_{\dot\gamma^*} \sum_{m=1}^{k-1} x_m \frac{\partial}{\partial\theta_m} = \sum_{m=1}^{k-1} \left(x_m \nabla_{\dot\gamma^*} \frac{\partial}{\partial\theta_m} + \dot x_m \frac{\partial}{\partial\theta_m} \right) = \sum_{m=1}^{k-1} \dot x_m \frac{\partial}{\partial\theta_m},$$

and similarly,

$$\nabla_{\dot\gamma^*} \nabla_{\dot\gamma^*} X = \sum_{m=1}^{k-1} \ddot x_m \frac{\partial}{\partial\theta_m}.$$

On the other hand, since R defined in (3.9) is a trilinear tensor, by expansion we have

$$R(\dot\gamma^*, X)\dot\gamma^* = \sum_{i,j,l,m=1}^{k-1} R^m_{ijl} x_j \frac{\partial}{\partial\theta_m}.$$

So the Jacobi equation (3.11) along γ^* is reduced to

$$\sum_{m=1}^{k-1} \left(\ddot x_m + \sum_{i,j,l=1}^{k-1} R^m_{ijl} x_j \right) \frac{\partial}{\partial\theta_m} = 0,$$

or equivalently, to the following group of second order differential equations on $\mathbb{R}^{k-1}$:

$$\ddot x_m + \sum_{i,j,l=1}^{k-1} R^m_{ijl} x_j = 0, \quad m = 1, \ldots, k-1. \tag{3.13}$$

Equations (3.13) can be written in matrix form as:

$$\ddot x + B_k x = 0, \tag{3.14}$$

where $B_k = (b_{mj})_{1\le m,j\le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ is a constant matrix whose component on the m-th row and j-th column is defined by

$$b_{mj} = \sum_{i,l=1}^{k-1} R^m_{ijl}.$$

In Appendix A we will prove the following simple expression for B_k.

Lemma 3 *Define* $\Lambda \triangleq \mathrm{diag}(\frac{k-1}{2}, \ldots, \frac{i(k-i)}{2}, \ldots, \frac{k-1}{2})_{1\le i\le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$. *Then*

$$B_k = \Delta^{-1}\Lambda - I_k, \tag{3.15}$$

where I_k *is the* $(k-1)$*-by-*$(k-1)$ *identity matrix.*

Remark 3 B_k *is a constant matrix independent of* t *since the metric of* N *is homogeneous along* γ^**, or more precisely, for each* $\tau > 0$*, the map* $(q_1, \ldots, q_k) \in N \mapsto (R_\tau(q_1), \ldots, R_\tau(q_k)) \in N$ *is an isometry of* N *mapping* $\gamma^*(t)$ *to* $\gamma^*(t+\tau)$ *whose differential map takes* $\frac{\partial}{\partial\theta_i}\big|_{\gamma^*(t)}$ *to* $\frac{\partial}{\partial\theta_i}\big|_{\gamma^*(t+\tau)}$ *for each* $i = 1, \ldots, k-1$.

The solutions of the Jacobi equation (3.14) are closely related to the spectral decomposition of B_k. To compute the eigenvalues of B_k, define a matrix

$$U \triangleq \begin{bmatrix} u_1 & \cdots & u_{k-1} \end{bmatrix}$$

whose column vectors u_j for $j = 1, \dots, k-1$ are given by

$$u_j = \left[\left(\frac{2-k}{k} \right)^{j-1}, \dots, \left(\frac{2i-k}{k} \right)^{j-1}, \dots, \left(\frac{k-2}{k} \right)^{j-1} \right]^T \in \mathbb{R}^{k-1}. \tag{3.16}$$

U is a Vandermonde matrix, hence nonsingular.

The following lemma can be verified directly.

Lemma 4 *For each $j = 1, \dots, k-1$,*

$$\Delta^{-1}\Lambda u_j = \frac{j(j+1)}{2} u_j + \textit{linear combination of } u_1, \dots, u_{j-1}. \tag{3.17}$$

In matrix form, (3.17) is equivalent to $\Delta^{-1}\Lambda U = U\Sigma$, i.e., $U^{-1}\Delta^{-1}\Lambda U = \Sigma$, where Σ is an upper triangular matrix whose elements on the main diagonal are $\frac{j(j+1)}{2}$, $j = 1, \dots, k-1$. Thus $\Delta^{-1}\Lambda$ and Σ have the same set of eigenvalues. As a result,

Corollary 2 *B_k has $k-1$ distinctive eigenvalues $\lambda_j = \frac{j(j+1)}{2} - 1$, for $j = 1, \dots, k-1$.*

Denote by v_j an eigenvector of $\Delta^{-1}\Lambda$ corresponding to the eigenvalue $\lambda_j + 1$, i.e.,

$$\Delta^{-1}\Lambda v_j = (\lambda_j + 1) v_j = \frac{j(j+1)}{2} v_j, \tag{3.18}$$

for $j = 1, \dots, k-1$. Then by (3.15) v_j is also an eigenvector of B_k corresponding to the eigenvalue λ_j. Lemma 4 and equation (3.16) imply that v_j is of the form

$$v_j = \left[P_j\left(\frac{2-k}{k} \right), \dots, P_j\left(\frac{2i-k}{k} \right), \dots, P_j\left(\frac{k-2}{k} \right) \right]^T \tag{3.19}$$

for some non-trivial polynomial P_j of degree $j-1$.

In Appendix B, we shall derive from equations (3.18) and (3.19) a general condition (B.3) on the polynomials P_j, and prove that P_j are the discrete version of some orthogonal polynomials called the *ultraspherical* (or *Gegenbauer*) polynomials $C_{j-1}^{(\alpha)}$ with parameter $\alpha = \frac{3}{2}$. In particular, P_j is an even polynomial when j is odd, and an odd polynomial when j is even [1].

Using condition (B.3), one can determine the first few P_j (up to a scaling factor):

$$P_1(x) = 1, \quad P_2(x) = 3x, \quad P_3(x) = 5x^2 - \left(1 - \frac{4}{k^2}\right), \quad P_4(x) = 7x^3 + \left(\frac{20}{k^2} - 3\right) x.$$

The first few v_j can then be obtained by (3.19). Therefore,

Lemma 5 *B_k has the following eigenvectors:*

- $v_1 = [1, \dots, 1]^T$ *for* $\lambda_1 = 0$;
- $v_2 = [2-k, \dots, 2i-k, \dots, k-2]^T$ *for* $\lambda_2 = 2$;
- $v_3 = [4k^2 - 20k + 24, \dots, 5(2i-k)^2 - k^2 + 4, \dots, 4k^2 - 20k + 24]^T$ *for* $\lambda_3 = 5$.

Remark 4 *B_k has an eigenvalue 0 with the corresponding eigenvector $[1,\ldots,1]^T$ as a consequence of the fact that $\dot{\gamma}^* = \frac{\partial}{\partial\theta_1} + \cdots + \frac{\partial}{\partial\theta_{k-1}}$ is parallel along γ^*.*

For large values of j, the expression of the polynomial P_j can be quite complicated. Fortunately, one can verify directly the following lemma.

Lemma 6 *B_k has the following eigenvectors:*
- $v_{k-2} = [(2-k)\binom{k}{1},\ldots,(-1)^{i+1}(2i-k)\binom{k}{i},\ldots,(-1)^k(k-2)\binom{k}{k-1}]^T$ *for* $\lambda_{k-2} = \frac{(k-1)(k-2)}{2} - 1$;
- $v_{k-1} = [\binom{k}{1},\ldots,(-1)^{i+1}\binom{k}{i},\ldots,(-1)^k\binom{k}{k-1}]^T$ *for* $\lambda_{k-1} = \frac{k(k-1)}{2} - 1$.

Lemma 5 and Lemma 6 together give the eigenvectors of B_k corresponding to a few of its smallest and largest eigenvalues.

The eigenvectors $v_1,\ldots,v_{k-1}$ of B_k form a basis of $\mathbb{R}^{k-1}$. We can then express x in this basis as $x = \sum_{j=1}^{k-1} y_j v_j$, so that the Jacobi equation (3.14) becomes

$$\ddot{y}_j + \lambda_j y_j = 0, \quad 1 \le j \le k-1.$$

Assume that X, hence x, vanishes at $t = 0$. Then $y_1(0) = \cdots = y_{k-1}(0) = 0$. Non-trivial solutions to the above equations with $y_j(0) = 0$, $1 \le j \le k-1$, are

$$\begin{aligned} &y_1(t) = c_1 t, \\ &y_j(t) = c_j \sin(t\sqrt{\lambda_j}), \quad j = 2,\ldots,k-1, \end{aligned}$$

for some constants $c_1,\ldots,c_{k-1}$ not identically zero. Conjugate points are encountered at those time epochs $\tau > 0$ where $y_j(\tau) = 0$ for all j. This is possible only if $c_1 = 0$ and τ is an integer multiple of $\pi/\sqrt{\lambda_j}$ for some $j \in \{2,\ldots,k-1\}$. Therefore,

Theorem 1 *The set of conjugate points along γ^* is*

$$\{\gamma^*(\tau) : \tau = m\pi/\sqrt{\lambda_j} \text{ for some } m \in \mathbb{N} \text{ and some } j \in \{2,\ldots,k-1\}\ ,\tau \le t_f\}.$$

The first conjugate point along γ^* occurs at time

$$\tau_k \triangleq \frac{\pi}{\sqrt{\lambda_{k-1}}} = \frac{\pi\sqrt{2}}{\sqrt{(k-2)(k+1)}}. \tag{3.20}$$

A geodesic is no longer distance-minimizing beyond its first conjugate point. Thus,

Corollary 3 *γ^* is not a solution to the OFC problem if $t_f > \tau_k$.*

Note that $\tau_k \sim \frac{1}{k}$ as $k \to \infty$. The result for the case $k = 3$ was first proved in [7].

Remark 5 *Traveling along a geodesic emitting from a fixed starting point, the last point for which the corresponding geodesic segment remains distance minimizing is called a* cut point *[4]. Along the geodesic a cut point is either the first conjugate point, or the first point that are connected by at least two distinct (possibly far way) geodesics to the starting point. In the latter case, the cut point is encountered before the first conjugate point. In our case we conjecture that the cut point along γ^* coincides with the first conjugate point, or in other words, γ^* is an optimal solution to the OFC problem for all t_f up to τ_k.*

3.3. Infinitesimal Perturbations beyond the Conjugate Points. We now show how shorter curves than γ^* with the same end points look, at least infinitesimally, once γ^* surpasses its conjugate points.

Let $\{\gamma_s^*\}_{-\epsilon<s<\epsilon}$ be a C^∞ proper variation of γ^* in N with variation field

$$X \triangleq \frac{\partial \gamma_s^*}{\partial s}\Big|_{s=0},$$

which is a vector field along γ^*. For each s, define $E(s)$ as the energy of γ_s^* in N, which coincides with the cost function of the joint trajectory corresponding to γ_s^*. By the variation of energy formulas [9], $E'(0)=0$ since γ^* is a geodesic, and

$$E''(0) = -\int_0^{t_f} \langle X, \nabla_{\dot\gamma^*}\nabla_{\dot\gamma^*} X + R(\dot\gamma^*, X)\dot\gamma^*\rangle\, dt.$$

Write $X = \sum_{i=1}^{k-1} x_i \frac{\partial}{\partial\theta_i}$ in the basis $\frac{\partial}{\partial\theta_1},\dots,\frac{\partial}{\partial\theta_{k-1}}$ along γ^*. Since $\{\gamma_s^*\}_{-\epsilon<s<\epsilon}$ is a proper variation, vector $x=(x_1,\dots,x_{k-1})$ vanishes at time 0 and t_f. In this coordinate system, the above equation reduces to

$$E''(0) = -\int_0^{t_f} x^T\Delta(\ddot x + B_k x)\, dt.$$

Suppose now that $t_f > \pi/\sqrt{\lambda_j}$ for some $j\in\{2,\dots,k-1\}$. Then, by choosing $\{\gamma_s^*\}_{-\epsilon<s<\epsilon}$ such that $x(t) = v_j\sin(\pi t/t_f)$ where we recall that v_j is an eigenvector of B_k associated with the eigenvalue λ_j, we have

$$E''(0) = -(\lambda_j - \frac{\pi^2}{t_f^2})(v_j^T\Delta v_j)\int_0^{t_f}\sin^2(\pi t/t_f)\, dt < 0, \tag{3.21}$$

since $v_j^T\Delta v_j > 0$ and $\lambda_j - \pi^2/t_f^2 > 0$. Therefore, γ_s^* is shorter than γ^* for sufficiently small s.

To sum up, the above analysis shows that if $t_f > \pi/\sqrt{\lambda_j}$ for some $j\in\{2,\dots,k-1\}$, a solution better than γ^* can be obtained by infinitesimally perturbing γ^* in such a way that, at each $t\in[0,t_f]$, $(\theta_1,\dots,\theta_{k-1})$ is incremented by an amount of $v_j\sin(\pi t/t_f)ds$. The linked-rod system (snake) formed by connecting successive agents will then assume a shape determined by the signs of the components of v_j. For example, the alternating signs of the components of v_{k-1} indicate a perturbation where the $k-1$ rods are first folded into a saw-like shape during the first half of the time interval $[0,t_f]$, with the degree of folding of each rod depending on its position (in fact, proportional to $\binom{k}{l}$ for the l-th rod from the edge, $l=1,\dots,k-1$), and then straightened up during the later half of the time interval. In contrast, v_2 indicates the $k-1$ rods to bend into a bow-like shape, whereas the shape specified by v_{k-2} is a mixing (product) of the bending specified by v_2 and the folding specified by v_{k-1}. The maximal perturbation occurs at $t=t_f/2$. Fig. 3.2 plots the various shapes of the linked-rod system at time $t=t_f/2$ caused by the perturbations $v_j\sin(\pi t/t_f)$ when $k=8$. Note that these shapes have been rotated to align with the x-axis.

The efficiency of the perturbations specified by different v_j, provided that $t_f > \pi/\sqrt{\lambda_j}$, can be studied by comparing the respective $E''(0)$ under the requirement that $\int_0^{t_f}\|X\|^2\,dt$ is constant. Since $\int_0^{t_f}\|X\|^2\,dt = (v_j^T\Delta v_j)\int_0^{t_f}\sin^2(\pi t/t_f)\,dt$, we can conclude by (3.21) that the larger the eigenvalue λ_j, the more efficient the perturbation specified by its corresponding eigenvector v_j. This implies that the most efficient perturbation is the one given by v_{k-1}.

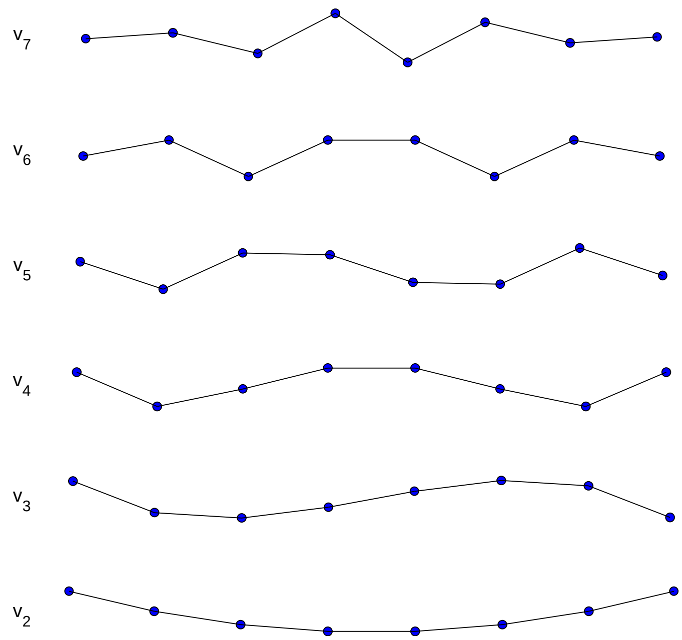

FIG. 3.2. *Perturbed shapes corresponding to* v_l*,* $l = 2, \dots, k-1$ *when* $k = 8$*.*

4. Conclusions. In this paper, we formulate the optimal formation constrained multi-agent coordination problem and give a geometric interpretation of its solutions as geodesics in a certain manifold. We study an instance of the problem and characterize analytically the conjugate points of a geodesic proposed as a candidate solution. We conclude that this geodesic is optimal for sufficiently close starting and destination positions, but no longer optimal after surpassing its first conjugate point, which occurs from the starting position at a distance that decreases to zero at the same rate as $1/k$ as the number k of agents increases. In this case, infinitesimally better coordinated motions are determined. We show that the analytical forms of these motions can be derived from a certain family of othogonal polynomials.

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Appendix A. Proof of Lemma 3.

In this proof, all terms in the equations, such as R^m_{ijl}, the Christoffel symbols and their derivatives, are evaluated along γ^*, for which $\theta_1 = \cdots = \theta_{k-1}$ and the conclusion of Lemma 2 holds. Thus (3.10) becomes

$$R^m_{ijl} = \frac{\partial \Gamma^m_{il}}{\partial \theta_j} - \frac{\partial \Gamma^m_{jl}}{\partial \theta_i}, \quad 1 \le i, j, l, m \le k-1,$$

and for each $1 \le m, j \le k-1$,

$$b_{mj} = \sum_{i,l} R^m_{ijl} = \sum_{i,l} \frac{\partial \Gamma^m_{il}}{\partial \theta_j} - \sum_{i,l} \frac{\partial \Gamma^m_{jl}}{\partial \theta_i} = \sum_i \frac{\partial \Gamma^m_{ii}}{\partial \theta_j} - \sum_i \frac{\partial \Gamma^m_{jj}}{\partial \theta_i}, \tag{A.1}$$

where the summations are all from 1 to $k-1$. The first term can be simplified to

$$\begin{aligned}\sum_i \frac{\partial \Gamma^m_{ii}}{\partial \theta_j} &= \sum_{i\neq j} \frac{\partial \Gamma^m_{ii}}{\partial \theta_j} + \frac{\partial \Gamma^m_{jj}}{\partial \theta_j} \\ &= \sum_{i\neq j} \frac{\partial}{\partial \theta_j}\left[\sum_l \Delta_{il}\sin(\theta_l-\theta_i) g^{lm}\right] + \frac{\partial}{\partial \theta_j}\left[\sum_{l\neq j} \Delta_{jl}\sin(\theta_l-\theta_j) g^{lm}\right] \\ &= \sum_{i\neq j} \Delta_{ij} g^{jm} - \sum_{l\neq j} \Delta_{jl} g^{lm},\end{aligned}$$

where we have used the fact that $\theta_1 = \cdots = \theta_{k-1}$ on γ^*. Similarly,

$$\begin{aligned}\sum_i \frac{\partial \Gamma^m_{jj}}{\partial \theta_i} &= \sum_{i\neq j} \frac{\partial}{\partial \theta_i}\left[\sum_l \Delta_{jl}\sin(\theta_l-\theta_j) g^{lm}\right] + \frac{\partial}{\partial \theta_j}\left[\sum_{l\neq j} \Delta_{jl}\sin(\theta_l-\theta_j) g^{lm}\right] \\ &= \sum_{i\neq j} \Delta_{ji} g^{im} - \sum_{l\neq j} \Delta_{jl} g^{lm}.\end{aligned}$$

Hence (A.1) can be rewritten as

$$b_{mj} = \sum_{i\neq j} \Delta_{ij} g^{jm} - \sum_{i\neq j} \Delta_{ji} g^{im} = (\sum_i \Delta_{ij}) g^{jm} - \sum_i \Delta_{ji} g^{im}.$$

Since $g_{ji} = \Delta_{ji}$ on γ^*, $\sum_i \Delta_{ji} g^{im} = \sum_i g_{ji} g^{im} = \delta_{mj}$ by the definition of g^{im}. Moreover, $\sum_i \Delta_{ij} = \sum_{i\leq j} \frac{i(k-j)}{2} + \sum_{i>j} \frac{(k-i)j}{2} = \frac{j(k-j)}{2}$. Therefore,

$$b_{mj} = \frac{j(k-j)}{2} g^{jm} - \delta_{mj} = \frac{j(k-j)}{2} g^{mj} - \delta_{mj}, \tag{A.2}$$

by the symmetry of g^{mj}. Note that, on γ^*, $(g^{mj})_{1\leq m,j\leq k-1} = [(g_{mj})_{1\leq m,j\leq k-1}]^{-1} = \Delta^{-1}$. So (A.2) is exactly the desired conclusion.

Appendix B. Computation of Polynomials P_j.

In this appendix we shall derive the expressions of the polynomials P_j in equation (3.19), and prove that they belong to a certain family of orthogonal polynomials.

Fix a $j \in \{1,\dots,k-1\}$. Recall that the polynomial P_j is defined in (3.19), where v_j is given in (3.18). By substituting (3.19) into (3.18) and using Lemma 1, we have

$$\begin{aligned}&-\frac{1}{2}(i-1)(k-i+1)P_j\left(\frac{2i-2-k}{k}\right) + i(k-i)P_j\left(\frac{2i-k}{k}\right) - \frac{1}{2}(i+1)(k-i-1) \\ &\quad P_j\left(\frac{2i+2-k}{k}\right) = \frac{j(j+1)}{2} P_j\left(\frac{2i-k}{k}\right),\end{aligned}$$

for $i = 1,\dots,k-1$. This is equivalent to

$$\nabla^2_{2/k}[(1-x^2)P_j(x)] = -j(j+1)P_j(x), \tag{B.1}$$

for $x = \frac{2i-k}{k}$, $i = 1,\dots,k-1$. Here, for $h>0$, ∇^2_h denotes the operator

$$\nabla^2_h[P(x)] = \frac{1}{h^2}[-2P(x) + P(x-h) + P(x+h)], \tag{B.2}$$

which is a difference approximate of $P''(x)$ with step size h. ∇_h^2 maps a polynomial $P(x)$ of degree $m \geq 2$ to a polynomial of degree $m-2$. Thus in (B.1) both sides are polynomials of degree $j-1 \leq k-2$. Since the equality (B.1) holds for $k-1$ distinct values of x, it holds for all x. To sum up, P_j satisfies the following condition:

$$\nabla_{2/k}^2[(1-x^2)P_j(x)] = -j(j+1)P_j(x), \quad \forall x \in \mathbb{R}. \tag{B.3}$$

Note that $\nabla_h^2[P(x)] \to P''(x)$ as $h \to 0$. Therefore, for large k, the above condition can be approximated by

$$(1-x^2)P_j''(x) - 4xP_j'(x) + (j-1)(j+2)P_j(x) = 0.$$

The polynomial solution to this differential equation is called the $(j-1)$-th *ultraspherical* (or *Gegenbauer*) *polynomial* $C_{j-1}^{(\alpha)}(x)$ with parameter $\alpha = \frac{3}{2}$ [1, pp. 781]. The polynomials $C_{j-1}^{(3/2)}(x)$ obtained for different $j \geq 1$ are orthogonal on $[-1,1]$ with respect to the weight function $1-x^2$, i.e., $\int_{-1}^{1}(1-x^2)C_l^{(3/2)}(x)C_j^{(3/2)}(x)\,dx = 0$ for $l \neq j$ (see [18]). Since $P_j(x)$ tends to $C_{j-1}^{(3/2)}(x)$ as $k \to \infty$, we expect that the polynomials $P_j(x)$, $j = 1, \ldots, k-1$, are "approximately" orthogonal with respect to $1-x^2$ on $[-1,1]$ as well. Indeed, a discrete version of this orthogonality condition holds:

$$\sum_{x=\frac{2i-k}{k},\, i=1,\ldots,k-1} (1-x^2)P_l(x)P_j(x) = 0, \quad \text{for } l, j \in \{1, \ldots, k-1\},\ l \neq j. \tag{B.4}$$

Condition (B.4) can be easily proved as follows. Observe that

$$\sum_{x=\frac{2i-k}{k},\, i=1,\ldots,k-1} (1-x^2)P_l(x)P_j(x) = \frac{8}{k^2} v_l^T \Lambda\, v_j.$$

Thus we need only to show that $v_l^T \Lambda\, v_j = 0$ for $l \neq j$, or equivalently, that $V^T \Lambda V$ is diagonal, where V is the matrix defined as $V = \begin{bmatrix} v_1 & \cdots & v_{k-1} \end{bmatrix}$. Equation (3.18) implies that $\Lambda V = \Delta V \Omega$ where $\Omega = \mathrm{diag}(\lambda_1 + 1, \ldots, \lambda_i + 1, \ldots, \lambda_{k-1} + 1)$. Thus $V^T \Lambda V = (V^T \Delta V)\Omega$. Note that the left hand side $V^T \Lambda V$ is a symmetric matrix, while the right hand side is the product of a symmetric matrix $V^T \Delta V$ and a diagonal matrix Ω with distinct diagonal elements. In order for them to be equal, we must have that both $V^T \Lambda V$ and $V^T \Delta V$ are diagonal matrices. The diagonality of $V^T \Lambda V$ implies condition (B.4), and the diagonality of $V^T \Delta V$ implies that the tangent vectors represented by v_j, $j = 1, \ldots, k-1$, at any point on γ^* are orthogonal.

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