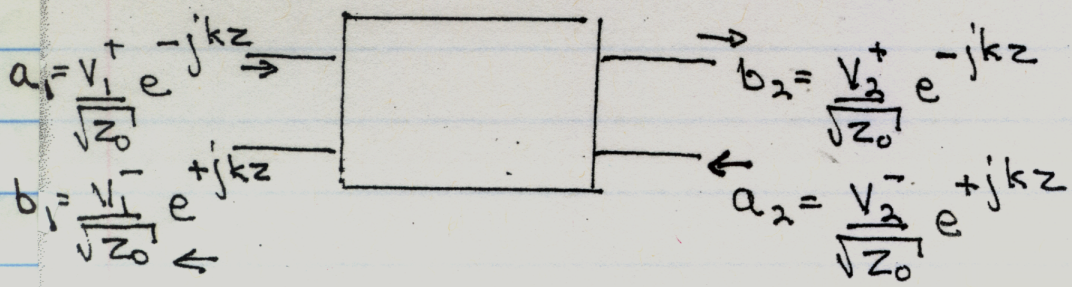
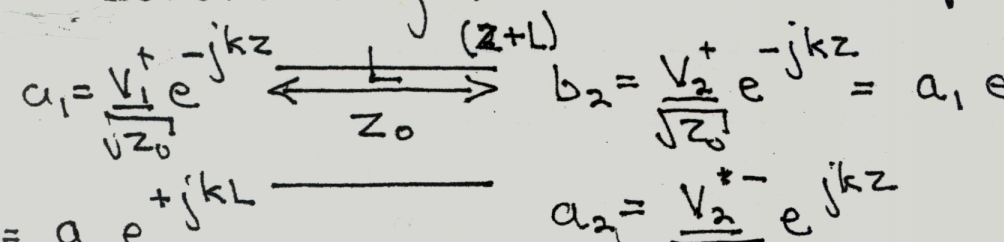
# Scattering parameters




## Scattering Parameters

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Short length of line - **Example 1**




Thus
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{jkL} \\ e^{-jkL} & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$S_{11}$ $S_{21}$ $S_{22}$ $S_{12}$

For lossless devices (e.g. couplers)

$$b_2 b_2^* - a_2 a_2^* = \text{net power out}$$
$$= a_1 a_1^* - b_1 b_1^* = \text{net power in}$$
$$b_2 b_2^* + b_1 b_1^* = a_1 a_1^* + a_2 a_2^*$$
$$(b_1^*, b_2^*) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = (a_1^*, a_2^*) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
$$(a_1^*, a_2^*) \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1^*, a_2^*) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$(S^{t*})(S) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \left| \begin{array}{l} S_{11} S_{11}^* + S_{21}^* S_{21} = 1 \\ S_{12}^* S_{11} + S_{22}^* S_{21} = 0 \\ S_{11}^* S_{12} + S_{21}^* S_{22} = 0 \end{array} \right|$$

reflection. This is a consequence of power conservation

Example (2b) <u>Resistor</u> (+ or −)


$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

make $\dfrac{1}{j\omega C Z_0} = \left(\dfrac{R}{Z_0}\right)$

$$S_{11} = -\frac{Z_0/R}{2 + Z_0/R} \quad ; \quad S_{22} = S_{11}$$

$$S_{12} = S_{21} = \frac{2}{2 + Z_0/R}$$

Note: Lack of Normalization (dissipation in R)

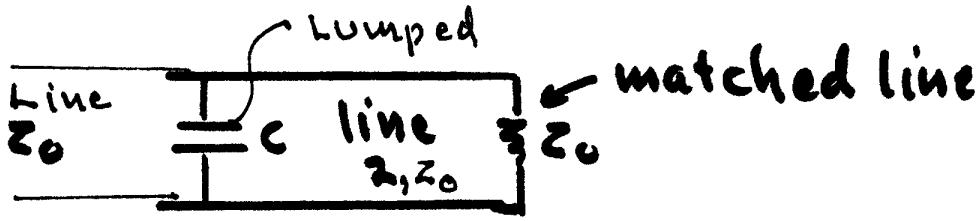
$$|S_{11}|^2 + |S_{12}|^2 = \frac{(Z_0/R)^2 + 4}{4 + (Z_0/R)^2 + 2 Z_0/R}$$

$< 1$ if $R > 0$

$> 1$ if $R < 0$ — gain

capacitor on a ~~the~~ matched line






$$b_1 = S_{11} a_1 + S_{12} a_2$$

Consider $a_1 \neq 0$ (incident wave at port 1)

$a_2 = 0$ since the line 2 is matched

$$\therefore \frac{b_1}{a_1} = S_{11} = \left(\frac{Z_L - Z_0}{Z_0 + Z_L}\right) = \frac{\frac{1}{j\omega C} \| Z_0 - Z_0}{\frac{1}{j\omega C} \| Z_0 + Z_0}$$

$$= \frac{-Z_0^2}{Z_0^2 + 2Z_0/j\omega C}$$

$$= \frac{-Z_0}{Z_0 j\omega C + 2} = S_{22} \text{ (by symmetry)}$$

$$b_2 = S_{21} a_1 = S a_1 \quad (a_2 = 0)$$

$$= \frac{2Z_L}{Z_L + Z_0} a_1 = 2 \frac{Z_0 \, 1/j\omega C}{Z_0 \, 1/j\omega C + Z_0 (1/j\omega C + Z_0)}$$

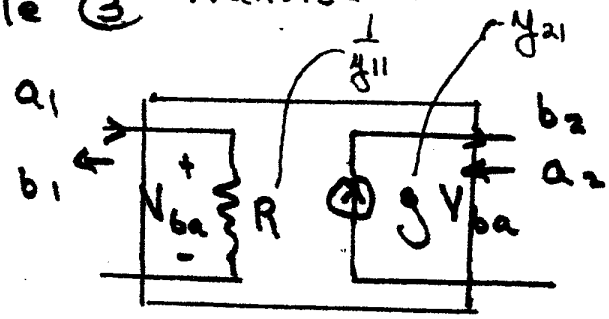
$$= \frac{2}{2 + j\omega C Z_0}$$

Capacitor is lossless so $S$ is unitary

$S_{12} = S_{21}$ (also because of symmetry)

Note: The relative $\pi$ phase shift of the transmission with respect to the

Example ③ Transistor




$y_{22} = 0$
$y_{12} = 0$

$$b_1 = \frac{(R - Z_0)}{R + Z_0} a_1 + 0\, a_2$$

$V_{ba} = (a_1 + b_1)\sqrt{Z_0}$ = voltage at the input

$b_2$ = normalized voltage = $Z_0$ (normalized current)

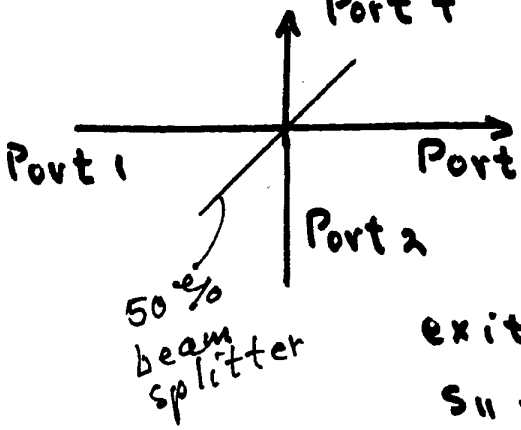
$$b_2 = Z_0 g\,(a_1 + b_1) = Z_0 g\,(a_1 + \rho a_1) = Z_0 g (1+\rho) a_1$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \rho & , & 0 \\ Z_0 g(1+\rho) & , & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Note $\rho = \frac{R - Z_0}{R + Z_0} = \frac{1/Y_{11} - Z_0}{1/Y_{11} + Z_0} = \frac{1 - Y_{11} Z_0}{1 + Y_{11} Z_0}$

Note that since the device is neither lossless nor reciprocal the S-matrix is neither symmetric nor unitary

For those interested in optics and quantum computing (outside scope of class) 




## The loss-less beam splitter
(For interest)

S is unitary

Field incident at port 1 $a_1$ exits port 3 and port 4

$$S_{11} = S_{22} = S_{33} = S_{44} = 0$$

$$b_1 = S_{11} a_1 = 0$$

$$b_2 = S_{21} a_1 = 0 \qquad S_{21} = 0$$

$$b_3 = S_{31} a_1 \qquad |S_{31}| = \frac{1}{\sqrt{2}}$$

$$b_4 = S_{41} a_1 \qquad |S_{41}| = \frac{1}{\sqrt{2}}$$

Power

$$b_3 b_3^* + b_4 b_4^* = (S_{31} S_{31}^* + S_{41} S_{41}^*) a_1 a_1^*$$

$$= a_1 a_1^*$$

Similarly for $a_2$ $\quad b_1 = 0 \Rightarrow S_{12} = 0$

$$b_3 = b_4 \Rightarrow |S_{32}| = |S_{42}| = \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} 0 & 0 & S_{13} & S_{14} \\ 0 & 0 & S_{23} & S_{24} \\ S_{31} & S_{32} & 0 & 0 \\ S_{41} & S_{42} & 0 & 0 \end{pmatrix}$$

input at 3 gives 0 at 4

Reciprocal Device

$S_{13} = S_{31} \quad S_{23} = S_{32}$

$S_{14} = S_{41} \quad S_{24} = S_{42}$

**Must be unitary**

$$S_{41} S_{31}^* + S_{32}^* S_{42} = 0$$

By symmetry $S_{32} = S_{41}$ and $S_{42} = S_{31}$

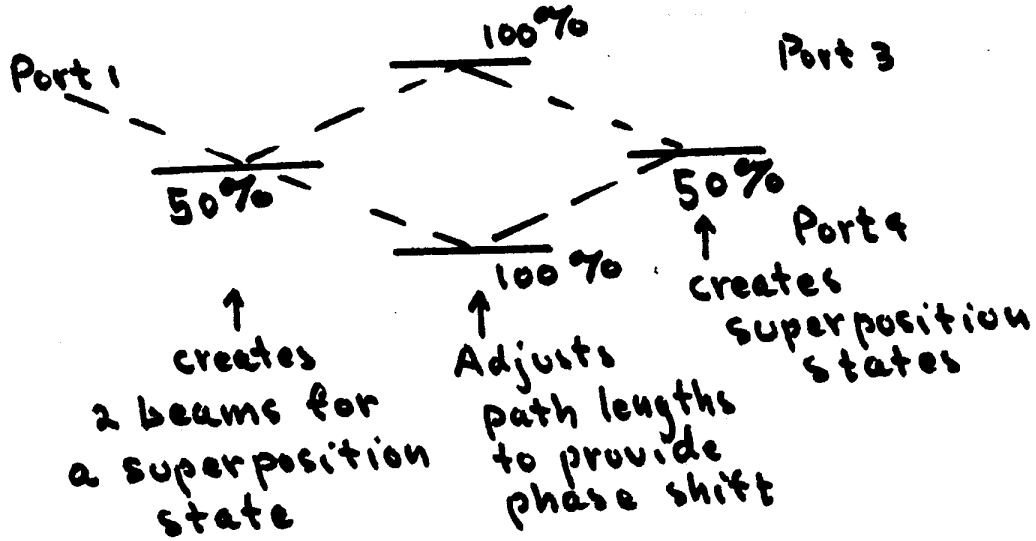
Thus this requires $S_{41} S_{42}^* + S_{41}^* S_{42} = 0$ (6)

$S_{41}$ is just the reflection $S_{42}$ the transmission. Thus $\Gamma \xi^* + \Gamma^* \xi = 0$

$|\Gamma| = |\xi| = \frac{1}{\sqrt{2}}$. The $e^{j\theta_\Gamma} e^{-j\theta_\xi} + c.c. = 0$

or $\cos(\theta_\Gamma - \theta_\xi) = 0 \Rightarrow \theta_\Gamma - \theta_\xi = \pi$ as with the capacitor.

This leads to the basic qubit based upon a pair of beam splitters.




For input at port 1 output (superposition state) can be at Port 3 or Port 4 or a combination depending upon the relative phase difference of the two paths