Scattering parameters

\[ a_1 = \frac{V_1^+ e^{-jkz}}{V_{01}^{\gamma}} \quad \rightarrow \quad b_1 = \frac{V_2^+ e^{-jkz}}{V_{02}^{\gamma}} \]

\[ b_1 = \frac{V_1^- e^{+jkz}}{V_{01}^{\gamma}} \quad \leftarrow \quad a_2 = \frac{V_2^- e^{+jkz}}{V_{02}^{\gamma}} \]

Scattering Parameters

\[ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \]

Short length of line - Example 1

\[ a_1 = \frac{V_1^+ e^{-jkz}}{V_{01}^{\gamma}} \quad \xrightarrow{(L+L)} \quad b_2 = \frac{V_2^+ e^{-jkz}}{V_{02}^{\gamma}} = a_1 e^{-jkL} \]

\[ b_1 = a_2 e^{+jkL} \quad \xrightarrow{(L+L)} \quad a_2 = \frac{V_2^- e^{+jkz}}{V_{02}^{\gamma}} = a_1 e^{-jkL} \]

Thus

\[ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{jkL} \\ e^{-jkL} & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \]

\[ \begin{pmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{pmatrix} \]

For lossless devices (e.g., couplers)

\[ b_2^* b_2 - a_2^* a_2 = \text{net power out} \]

\[ = a_1 a_1^* - b_1 b_1^* = \text{net power in} \]

\[ b_2 b_2^* + b_1 b_1^* = a_1 a_1^* + a_2 a_2^* \]

\[ (b_1^*, b_2^*) (b_1, b_2) = (a_1^*, a_2^*) (a_1, a_2) \]

\[ (a_1^*, a_2^*) (S_{11} S_{21}^* S_{12}^*) (S_{11} S_{12}) (a_1, a_2) = (a_1^*, a_2^*) (a_1, a_2) \]

\[ (S^{*}) (s) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} S_{11} + S_{21}^* S_{21} & 0 \\ S_{12} S_{11} + S_{22}^* S_{21} & S_{22} \end{pmatrix} = 0 \]

\[ S_{11} S_{22} + S_{21}^* S_{21} = 0 \]

\[ S_{11} S_{12} + S_{21}^* S_{22} = 0 \]
reflection. This is a consequence of power conservation

Example \( (ab) \) \textbf{Resistor \ (+ or -)}

\[ \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \]

make \[ \frac{1}{j\omega C z_0} \cdot \frac{R}{z_0} \]

\[ s_{11} = \frac{Z_0 R}{2 + Z_0 R} \quad ; \quad s_{22} = s_{11} \]

\[ s_{12} = s_{21} = \frac{2}{2 + Z_0 R} \]

Note: \textbf{Lack of Normalization (dissipation in R)}

\[ s_{11}^2 + |s_{12}|^2 = \frac{(Z_0 R)^2 + R}{4 + (Z_0 R)^2 + 2 Z_0 R} \]

\[ < 1 \text{ if } R > 0 \]

\[ > 1 \text{ if } R < 0 - \text{gain} \]
Capacitor on a line matched line

\[ Z_0 \]  

\[ \text{Line} \quad Z_{01} = \frac{1}{j\omega L} \]

Consider \( a_1 = 0 \) (incident wave at port 1)

\[ a_2 = 0 \quad \text{since the line 2 is matched} \]

\[ \frac{b_1}{a_1} = s_{11} = \frac{Z_0 - Z_L}{Z_0 + Z_L} = \frac{1}{j\omega L} \frac{1}{\frac{1}{j\omega L} Z_0 + Z_L} \]

\[ = - \frac{Z_0}{Z_0^2 + 2Z_0/j\omega} \]

\[ = \frac{Z_0}{Z_0/j\omega + 2} = s_{22} \]

\[ b_2 = s_{21} a_1 = \frac{3}{2} a_1 \quad (a_2 = 0) \]

\[ = \frac{2Z_0}{Z_0 + j\omega} \frac{a_1}{j\omega} \frac{Z_0}{2Z_0 + j\omega + Z_0(\frac{1}{j\omega} + Z_0)} \]

\[ = \frac{2}{2 + j\omega Z_0} \]

Capacitor is lossless so \( s \) is unitary

\[ s_{12} = s_{21} \quad \text{(also because of symmetry)} \]

Note: The relative \( \pi \) phase shift of the transmission with respect to the
Example 2: Transistor

\[ b_1 = \frac{(R-Z_0)}{R+Z_0} a_1 + 0 a_2 \]

\[ V_{ba} = (a_1 + b_1) \sqrt{Z_0} = \text{voltage at the input} \]

\[ b_2 = \text{normalized voltage} = Z_0 (\text{normalized current}) \]

\[ b_2 = Z_0 g \begin{pmatrix} a_1 + b_1 \end{pmatrix} = Z_0 g \begin{pmatrix} a_1 + g a_1 \end{pmatrix} = Z_0 g (1+g) a_1 \]

\[
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} g & 0 \\ Z_0 g (1+g) & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}
\]

Note: \[ g = \frac{R-Z_0}{R+Z_0} = \frac{X_{\text{eff}} - Z_0}{X_{\text{eff}} + Z_0} = \frac{1 - Y_{11} Z_0}{1 + Y_{11} Z_0} \]

Note that since the device is neither lossless nor reciprocal the S-matrix is neither symmetric nor unitary.
For those interested in optics and quantum computing (outside scope of class)

For interest

The loss-less beam splitter

Port 1

Port 2

Port 3

Port 4

S is unitary

Field incident at port 1 a_i exits port 3 and port 4

\[ s_{11} = s_{22} = s_{33} = s_{44} = 0 \]

\[ b_1 = s_{11} a_i = 0 \]

\[ b_2 = s_{21} a_i = 0 \]

\[ b_3 = s_{31} a_i \]

\[ b_4 = s_{41} a_i \]

\[ |b_3| = \frac{1}{ \sqrt{2}} \]

\[ |b_4| = \frac{1}{ \sqrt{2}} \]

Power

\[ b_3 b_3^* + b_4 b_4^* = (s_{31} s_{31}^* + s_{41} s_{41}^*) a_i a_i^* \]

\[ = a_i a_i^* \]

Similarly for a_i

\[ b_1 = 0 \Rightarrow s_{11} = 0 \]

\[ b_2 = 0 \Rightarrow 15321 = 15431 = \frac{1}{ \sqrt{2}} \]

\[ \begin{pmatrix}
0 & 0 & s_{13} & s_{14} \\
0 & 0 & s_{23} & s_{24} \\
S_{31} & S_{32} & 0 & 0 \\
S_{41} & S_{42} & 0 & 0
\end{pmatrix} \text{ Reciprocal Device} \]

\[ s_{13} = 3 s_{31} \]

\[ s_{23} = s_{32} \]

\[ s_{14} = 6 q \]

\[ s_{24} = s_{42} \]

Input at 3 gives 0 at 4

Must be unitary

\[ s_{41} s_{31}^* + s_{32} s_{42}^* = 0 \]

By symmetry \[ s_{32} = s_{41} \] and \[ s_{42} = s_{31} \]
Thus this requires $S_{41} S_{42} \ast S_{41} S_{42} \ast (6)$

$S_{41}$ is just the reflection $S_{42}$ the transmission. Thus $\Pi \Phi_q^* + \Pi^* \Phi_q = 0$

$|\Pi| = |\Phi_q| = \frac{1}{\sqrt{2}}$. The $\exp(i \Phi_q - j \Phi_q) + \text{c.c.} = 0$

or $\cos(\Phi_q - \Phi^*) \ast 0 \Rightarrow \Phi_q - \Phi^* = \pi$ as with the capacitor.

This leads to the basic qubit based upon a pair of beam splitters.

For input at Port 1 output (superposition state) can be at Port 2 or Port 4 or a combination depending upon the relative phase difference of the two paths.