EECS 219C: Formal Methods

Boolean Satisfiability Solving

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The Boolean Satisfiability Problem (SAT)

- Given:

  A Boolean formula $F(x_1, x_2, x_3, \ldots, x_n)$

- Can F evaluate to 1 (true)?
  - Is F satisfiable?
  - If yes, return values to x_i's (satisfying assignment) that make F true

Why is SAT important?

- Theoretical importance:
  - First NP-complete problem (Cook, 1971)
- Many practical applications:
  - Model Checking
  - Automatic Test Pattern Generation
  - Combinational Equivalence Checking
  - Planning in AI
  - Automated Theorem Proving
  - Software Verification
  - …

Terminology

- Variable, Literal
- Operators: AND, OR, NOT
- Clause, Cube/Monomial
- Conjunctive Normal Form (CNF)
- Disjunctive Normal Form (DNF)

An Example

- Inputs to SAT solvers are usually represented in CNF

$$(a + b + c)\ (a' + b' + c)\ (a + b' + c')\ (a' + b + c')$$

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$$(a + b + c)\ (a' + b' + c)\ (a + b' + c')\ (a' + b + c')$$

Why CNF?

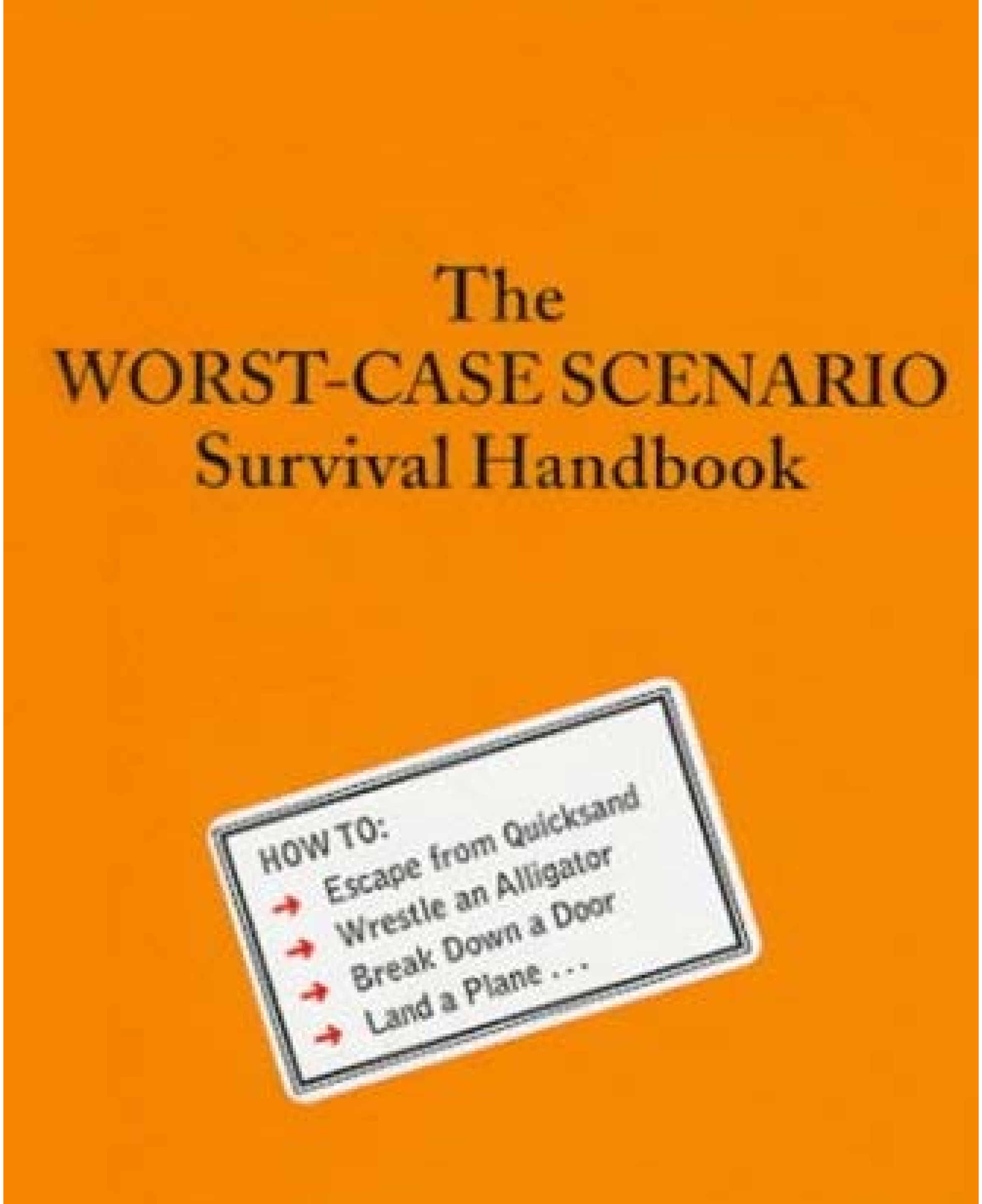
Why CNF?

- Input-related reason
  - Can transform from circuit to CNF in linear time & space (HOW?)
- Solver-related: Most SAT solver variants can exploit CNF
  - Easy to detect a conflict
  - Easy to remember partial assignments that don't work (just add 'conflict' clauses)
  - Other "ease of representation" points?
- Any reasons why CNF might NOT be a good choice?
  - Loses structural information present in circuits
  - Some relations, such as XOR, might be better exposed to the solver

Complexity of k-SAT

- A SAT problem with input in CNF with at most k literals in each clause
- Complexity for non-trivial values of k:
  - 2-SAT: in P
  - 3-SAT: NP-complete
  - > 3-SAT: ?

Worst-Case Complexity



Beyond Worst-Case Complexity

- What we really care about is “typical-case” complexity
- But how can one measure “typical-case”?
- Two approaches:
  - Is your problem a restricted form of 3-SAT? That might be polynomial-time solvable
  - Experiment with (random/domain-specific) SAT instances and analyze solver run-time vs formula parameters (#vars, #clauses, …)

Special Cases of 3-SAT that are polynomial-time solvable

- 2-SAT
  - T. Larrabee observed that many clauses in ATPG tend to be 2-CNF
- Horn-SAT
  - A clause is a Horn clause if at most one literal is positive
  - If all clauses are Horn, then problem is Horn-SAT
  - E.g. Application:- Checking that one finite-state system refines (implements) another

2-SAT Algorithm

- Linear-time algorithm (Aspvall, Plass, Tarjan, 1979)
  - Think of clauses as implications
  - Think of a graph with literals as nodes
  - Find strongly connected components
  - Variable and its negation should not be in the same component

- Example 1:

  (a’ + b) (b’ + c) (c’ + a)

- Example 2:

  (a’ + b) (b’ + c) (c’ + a) (a + b) (a’ + b’)

Horn-SAT

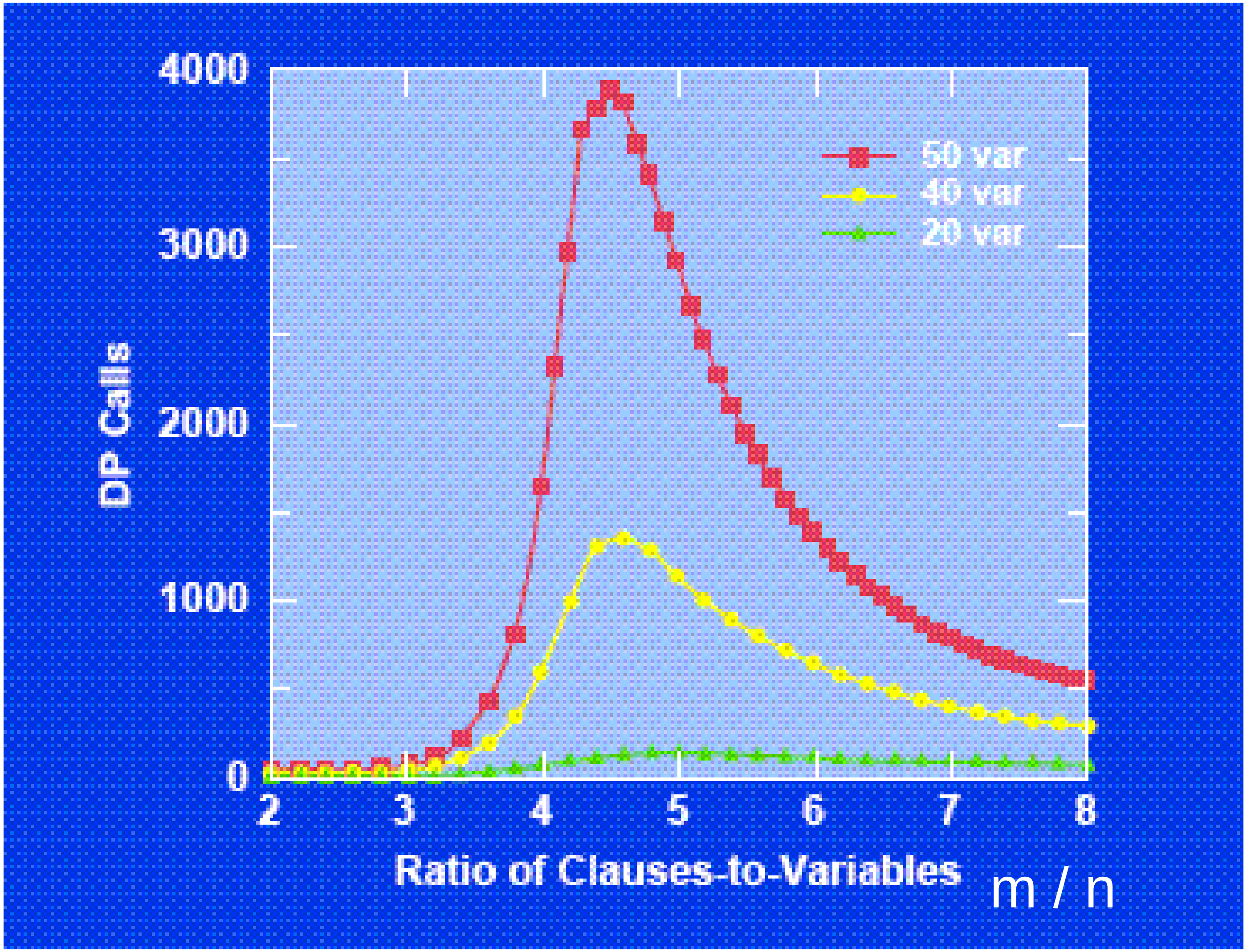
- Can we solve Horn-SAT in polynomial time? How? [homework]
  - Hint: again view clauses as implications.

- Variants:
  - Negated Horn-SAT: Clauses with at most one literal negative
  - Renamable Horn-SAT: Doesn't look like a Horn-SAT problem, but turns into one when polarities of some variables are flipped

Phase Transitions in k-SAT

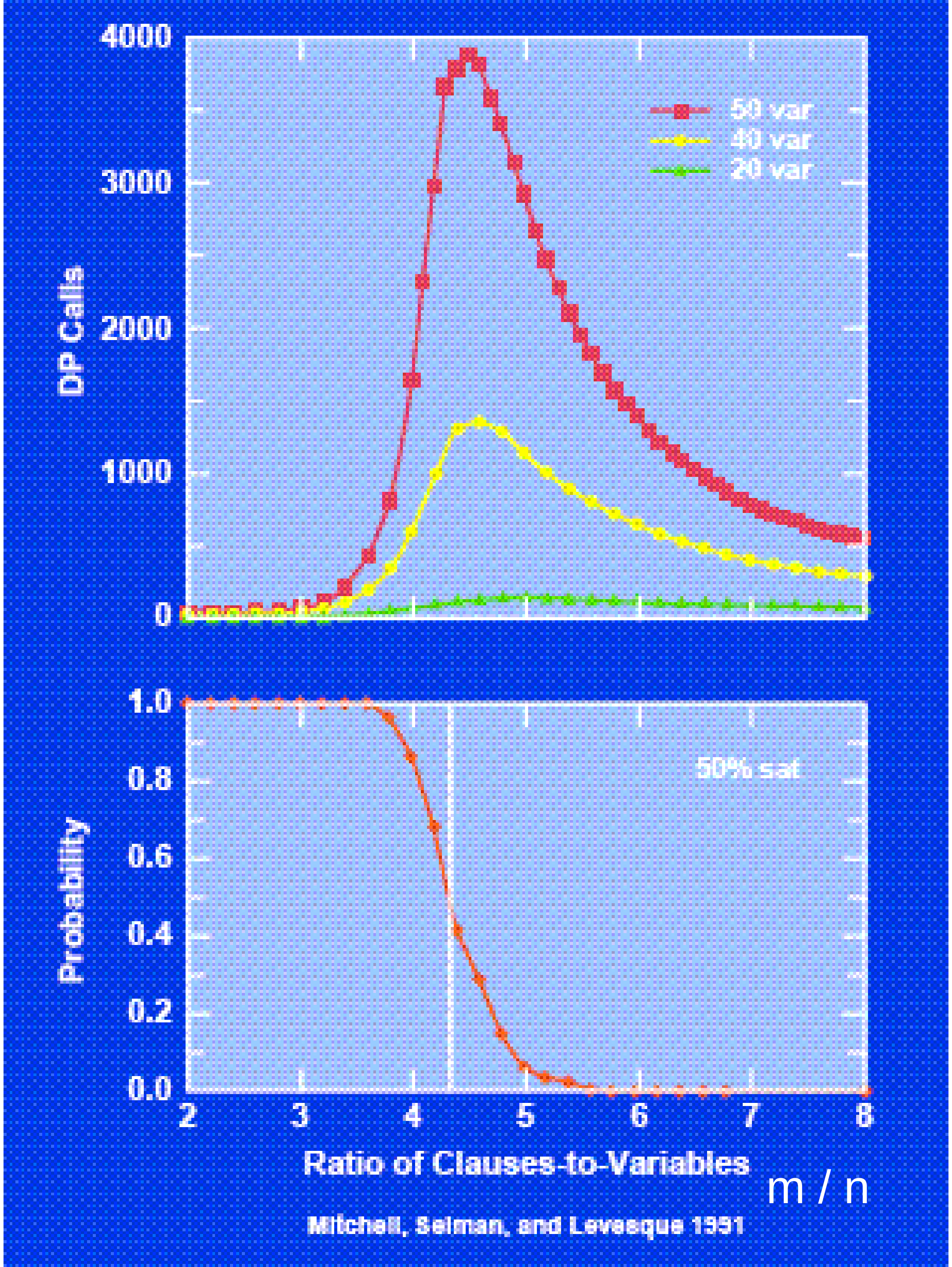
- Consider a fixed-length clause model
  - k-SAT means that each clause contains exactly k literals
- Let SAT problem comprise **m** clauses and **n** variables
  - Randomly generate the problem for fixed k and varying m and n
- Question: How does the problem hardness vary with m/n ?

3-SAT Hardness



As n increases hardness transition grows sharper

Transition at m/n ~ 4.3



Threshold Conjecture

- For every k, there exists a c* such that
  - For $m/n < c^*$, as $n \rightarrow \infty$, problem is satisfiable with probability 1
  - For $m/n > c^*$, as $n \rightarrow \infty$, problem is unsatisfiable with probability 1
- Conjecture proved true for k=2 and c*=1
- For k=3, current status is that c* is in the range 3.42 – 4.51

The (2+p)-SAT Model

- We know:
  - 2-SAT is in P
  - 3-SAT is in NP
- Problems are (typically) a mix of binary and ternary clauses
  - Let p $\in$ [0,1]
  - Let problem comprise (1-p) fraction of binary clauses and p of ternary
  - So-called (2+p)-SAT problem

Experimentation with random (2+p)-SAT

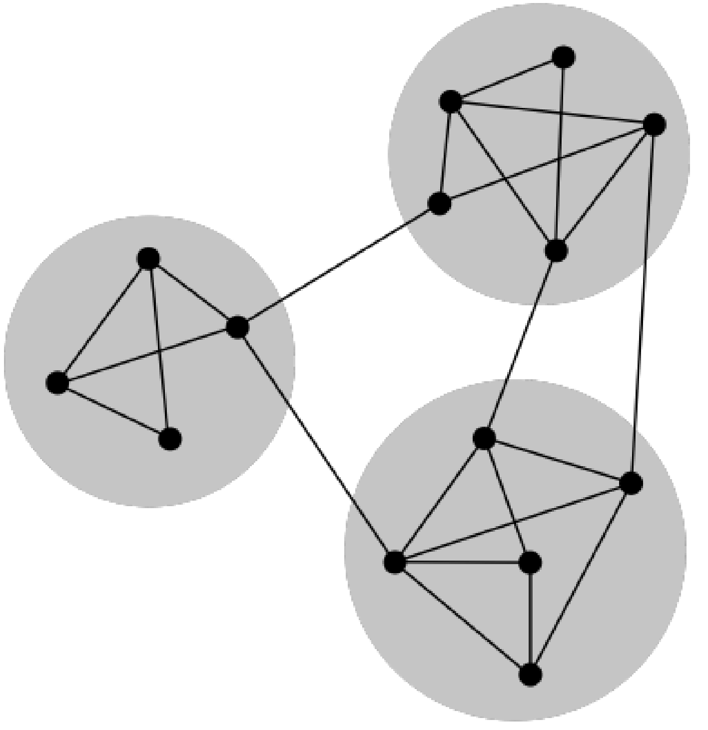
- When $p < \sim 0.41$
  - Problem behaves like 2-SAT
  - Linear scaling
- When $p > \sim 0.42$
  - Problem behaves like 3-SAT
  - Exponential scaling

- Intruiging observations, but don't help us predict behavior of problems in practice

Backbones and Backdoors

- Backbone [Parkes; Monasson et al.]
  - Subset of literals that must be true in every satisfying assignment (if one exists)
  - Empirically related to hardness of problems
- Backdoor [Williams, Gomes, Selman]
  - Subset of variables such that once you've given those a suitable assignment (if one exists), the rest of the problem is poly-time solvable
  - Also empirically related to hardness
- But no easy way to find such backbones / backdoors! ☹

Community Structure



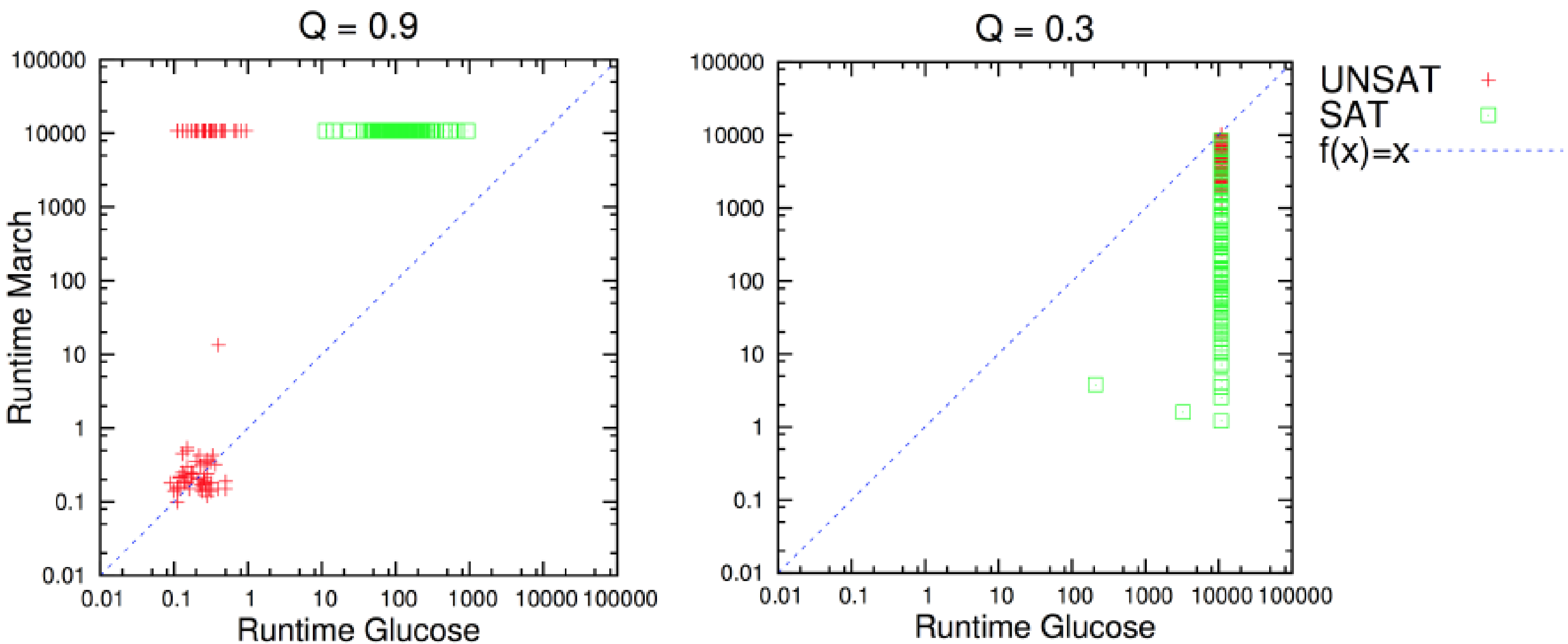
Intuition: Small, dense, weakly interconnected communities.

Community structure is typically measured by **modularity** (**Q-value**).

Empirical evidence relating modularity to CDCL performance: [Ansótegui et al., SAT 2012], [Newsham et al., SAT 2014], [Giráldez-Cru & Levy, IJCAI 2015].

Community Structure

Glucose (CDCL) is faster than March (look-ahead) on high modularity formulas, while March is faster on low modularity formulas. [Giráldez-Cru & Levy, IJCAI 2015].



However, this is not the whole story: see Mull, Fremont, & Seshia, SAT 2016 paper

A Classification of SAT Algorithms

- Davis-Putnam (DP)
  – Based on **resolution**
- Davis-Logemann-Loveland (DLL/DPLL)
  – Search-based
  – Basis for current most successful solvers (CDCL)
- Stalmarck's algorithm
  – More of a "breadth first" search, proprietary algorithm
- Stochastic search
  – Local search, hill climbing, etc.
  – Unable to prove unsatisfiability (incomplete)

Resolution

- Two CNF clauses that contain a variable x in opposite phases (polarities) imply a new CNF clause that contains all literals except x and x’

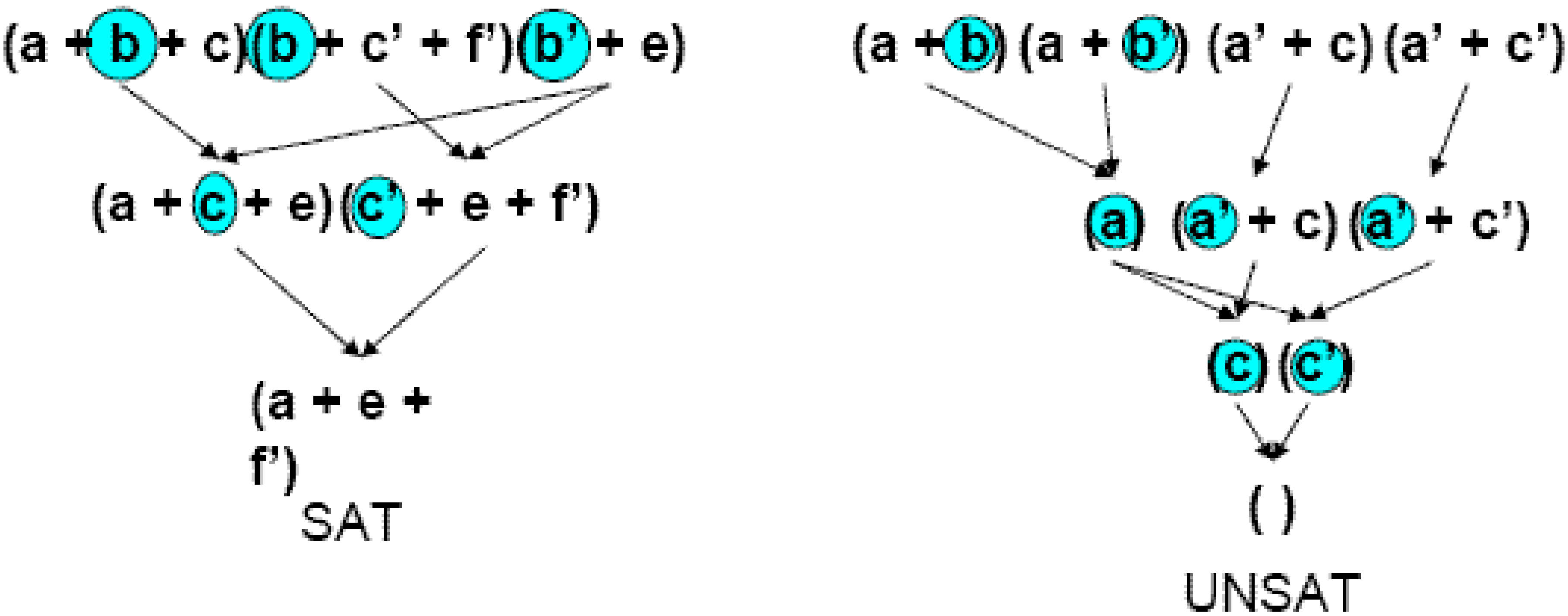
  $(a + b)\ (a' + c) = (a + b)(a' + c)(b + c)$

- Why is this true?

The Davis-Putnam Algorithm

- Iteratively select a variable x to perform resolution on
- Retain only the newly added clauses and the ones not containing x
- Termination: You either
  - Derive the empty clause (conclude UNSAT)
  - Or all variables have been selected

Resolution Example



How many clauses can you end up with?
(at any iteration)

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DLL Algorithm: General Ideas

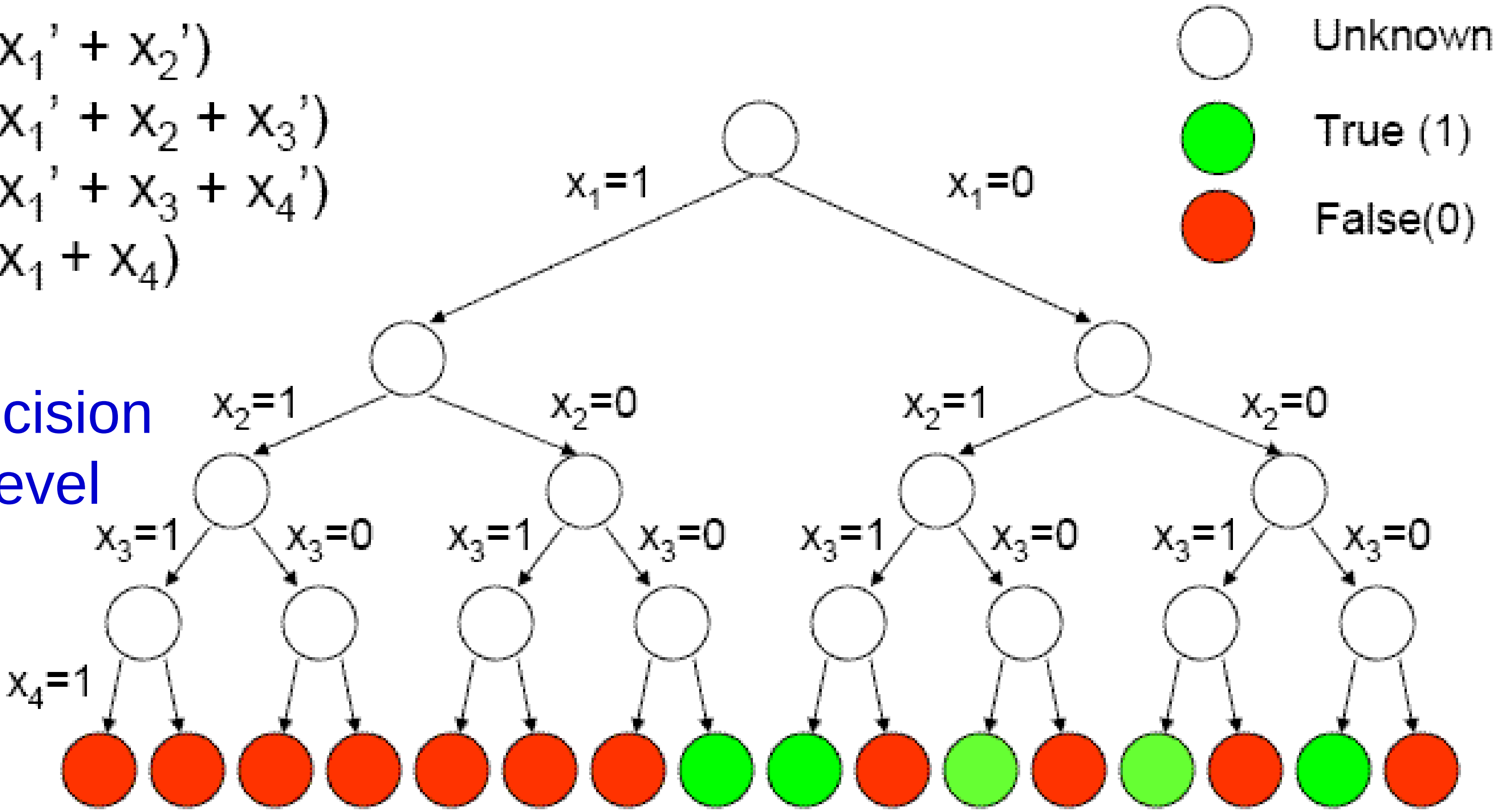
- Iteratively set variables until
  - you find a satisfying assignment (done!)
  - you reach a conflict (backtrack and try different value)
- Two main rules:
  - *Unit Literal Rule*: If an unsatisfied clause has all but 1 literal set to 0, the remaining literal must be set to 1

    (a + b + c) (d' + e) (a + c' + d)
  - *Conflict Rule*: If all literals in a clause have been set to 0, the formula is unsatisfiable along the current assignment path

Search Tree

$(x_1' + x_2')$
$(x_1' + x_2 + x_3')$
$(x_1' + x_3 + x_4')$
$(x_1 + x_4)$

Decision level



DLL Example 1

DLL Algorithm Pseudo-code

Preprocess

Branch

Propagate implications of that branch and deal with conflicts

```
DLL_iterative()
{
    status = preprocess();
    if (status!=UNKNOWN)
        return status;
    while(1) {
        decide_next_branch();
        while (true)
        {
            status = deduce();
            if (status == CONFLICT)
            {
                blevel = analyze_conflict();
                if (blevel < 0)
                    return UNSATISFIABLE;
                else
                    backtrack(blevel);
            }
            else if (status == SATISFIABLE)
                return SATISFIABLE;
            else break;
        }
    }
}
```

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```

Main Steps:

Pre-processing

Branching

Unit propagation (apply unit rule)

Conflict Analysis & Backtracking

Pre-processing: Pure Literal Rule

- If a variable appears in only one phase throughout the problem, then you can set the corresponding literal to 1
  - E.g. if we only see a’ in the CNF, set a’ to 1 (a to 0)
- Why?

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Conflicts & Backtracking

- Chronological Backtracking
  - Proposed in original DLL paper
  - Backtrack to highest (largest) decision level that has not been tried with both values
    - But does this decision level have to be the reason for the conflict?

Non-Chronological Backtracking

- Jump back to a decision level “higher” than the last one
- Also combined with “conflict-driven learning”
  - Keep track of the reason for the conflict
- Proposed by Marques-Silva and Sakallah in 1996
  - Similar work by Bayardo and Schrag in ‘97

DLL Example 2

DLL Algorithm Pseudo-code

```
DLL_iterative()
{
    status = preprocess();
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```

Main Steps:

Pre-processing

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Unit propagation (apply unit rule)

Conflict Analysis & Backtracking

Branching

- Which variable (literal) to branch on (set)?
- This is determined by a “decision heuristic”

- What makes a “decision heuristic” good?

Decision Heuristic Desiderata

- If the problem is satisfiable
  - Find a short partial satisfying assignment
  - GREEDY: If setting a literal will satisfy many clauses, it might be a good choice
- If the problem is unsatisfiable
  - Reach conflicts quickly (rules out bigger chunks of the search space)
  - Similar to above: need to find a short partial falsifying assignment
- Also: Heuristic must be cheap to compute!

Sample Decision Heuristics

- RAND
  - Pick a literal to set at random
  - What's good about this? What's not?
- Dynamic Largest Individual Sum (DLIS)
  - Let cnt(l) = number of occurrences of literal l in unsatisfied clauses
  - Set the l with highest cnt(l)
  - What's good about this heuristic?
  - Any shortcomings?

DLIS: A Typical Old-Style Heuristic

- Advantages
  - Simple to state and intuitive
  - Targeted towards satisfying many clauses
  - Dynamic: Based on current search state
- Disadvantages
  - Very expensive!
  - Each time a literal is set, need to update counts for all other literals that appear in those clauses
  - Similar thing during backtracking (unsetting literals)
- Even though it is dynamic, it is “Markovian” – somewhat static
  - Is based on current state, without any knowledge of the search path to that state

VSIDS: The Chaff SAT solver heuristic

- Variable State Independent Decaying Sum
  - For each literal l, maintain a VSIDS score
  - Initially: set to cnt(l)
  - Increment score by 1 each time it appears in an added (conflict) clause
  - Divide all scores by a constant (2) periodically (every N backtracks)
- Advantages:
  - Cheap: Why?
  - Dynamic: Based on search history
  - Steers search towards variables that are common reasons for conflicts (and hence need to be set differently)

Key Ideas so Far

- Data structures: Implication graph
- Conflict Analysis: Learn (using cuts in implication graph) and use non-chronological backtracking
- Decision heuristic: must be dynamic, low overhead, quick to conflict/solution

- Principle: Keep #(memory accesses)/step low
  - A step → a primitive operation for SAT solving, such as a branch

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Main Steps:

Pre-processing

Branching

Unit propagation
(apply unit rule)

Conflict Analysis
& Backtracking

Unit Propagation

- Also called Boolean constraint propagation (BCP)
- Set a literal and propagate its implications
  - Find all clauses that become unit clauses
  - Detect conflicts
- Backtracking is the reverse of BCP
  - Need to unset a literal and 'rollback'
- In practice: Most of solver time is spent in BCP
  - Must optimize!

BCP

- Suppose literal l is set. How much time will it take to propagate just that assignment?
- How do we check if a clause has become a unit clause?
- How do we know if there's a conflict?

- Introductory BCP slides

Detecting when a clause becomes unit

- Watch only two literals per clause. Why does this work?
- If one of the watched literals is assigned 0, what should we do?
- A clause has become unit if
  - Literal assigned 0 *must* continue to be watched, other watched literal unassigned
- What if other watched literal is 0?
- What if a watched literal is assigned 1?

- Lintao’s BCP example

2-literal Watching

- In a L-literal clause, $L \geq 3$, which 2 literals should we watch?

Comparison:

Naïve 2-counters/clause vs 2-literal watching

- When a literal is set to 1, update counters for all clauses it appears in
- No need for update
- Same when literal is set to 0
- Update watched literal
- If a literal is set, need to update each clause the variable *appears* in
- If a literal is set to 0, need to update only each clause it is *watched* in
- During backtrack, must update counters
- No updates needed during backtrack! (why?)

Overall effect: Fewer clauses accesses in 2-lit

zChaff Relative Cache Performance

		1dlx_c_mc_ex_bp_f		Hanoi4	
		Num Access	Miss Rate	Num Access	Miss Rate
Z-Chaff	L1	24,029,356	4.75%	364,782,257	5.38%
	L2	1,659,877	4.63%	30,396,519	11.65%
SATO (-g100)	L1	188,352,975	36.76%	465,160,957	41.76%
	L2	79,422,894	9.74%	202,495,679	16.77%
Grasp	L1	415,572,501	32.89%	876,250,978	32.53%
	L2	153,490,555	50.25%	335,713,542	51.15%

The programs are compiled with –O3 using g++ 2.8.1(for GRASP and Chaff) or gcc 2.8.1 (for Sato3.2.1) on Sun OS 4.1.2 Trace was generated with QPT quick tracing and profiling tool. Trace was processed with dineroIV, the memory configuration is similar to a Pentium III processor:

L1: 16K Data, 16K Instruction, L2: 256k Unified. Both have 32 byte cache line, 4 way set associativity.

Key Ideas in Modern DLL SAT Solving

- Data structures: Implication graph
- Conflict Analysis: Learn (using cuts in implication graph) and use non-chronological backtracking
- Decision heuristic: must be dynamic, low overhead, quick to conflict/solution
- Unit propagation (BCP): 2-literal watching helps keep memory accesses down

- Principle: Keep #(memory accesses)/step low
  - A step → a primitive operation for SAT solving, such as a branch

Other Techniques

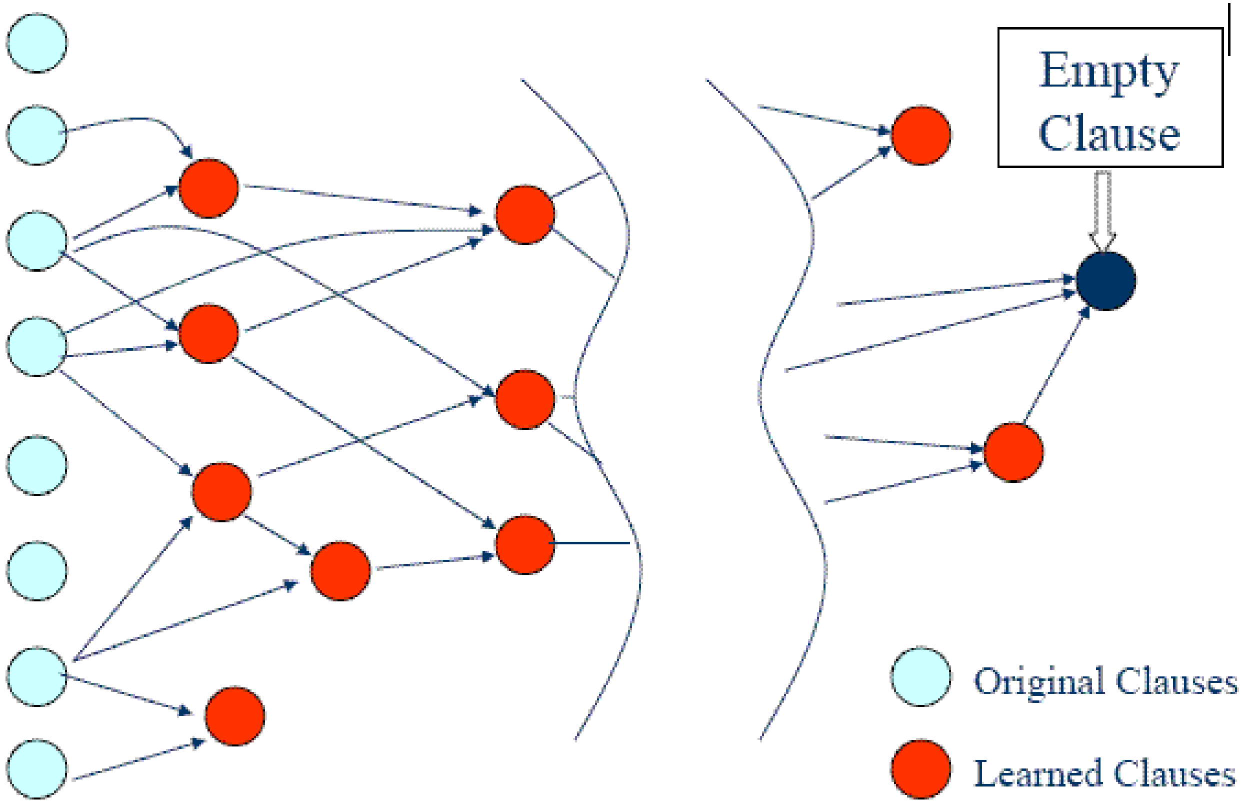
- Random Restarts
  - Periodically throw away current decision stack and start from the beginning
    - Why will this change the search on restart?
  - Used in most modern SAT solvers
- Clause deletion
  - Conflict clauses take up memory
    - What's the worst-case blow-up?
  - Delete periodically based on some heuristic ("age", length, etc.)
- Preprocessing/"Inprocessing" and Rewriting techniques give a lot of performance improvements in recent solvers

Proof

- Starting from facts (clauses), the SAT solver has presumably derived "unsatisfiable" (the empty clause)
- So there must be a way of going step-by-step from input clauses to the empty clause using rules
  - In fact, there's only one rule: **resolution**

Resolution Graph

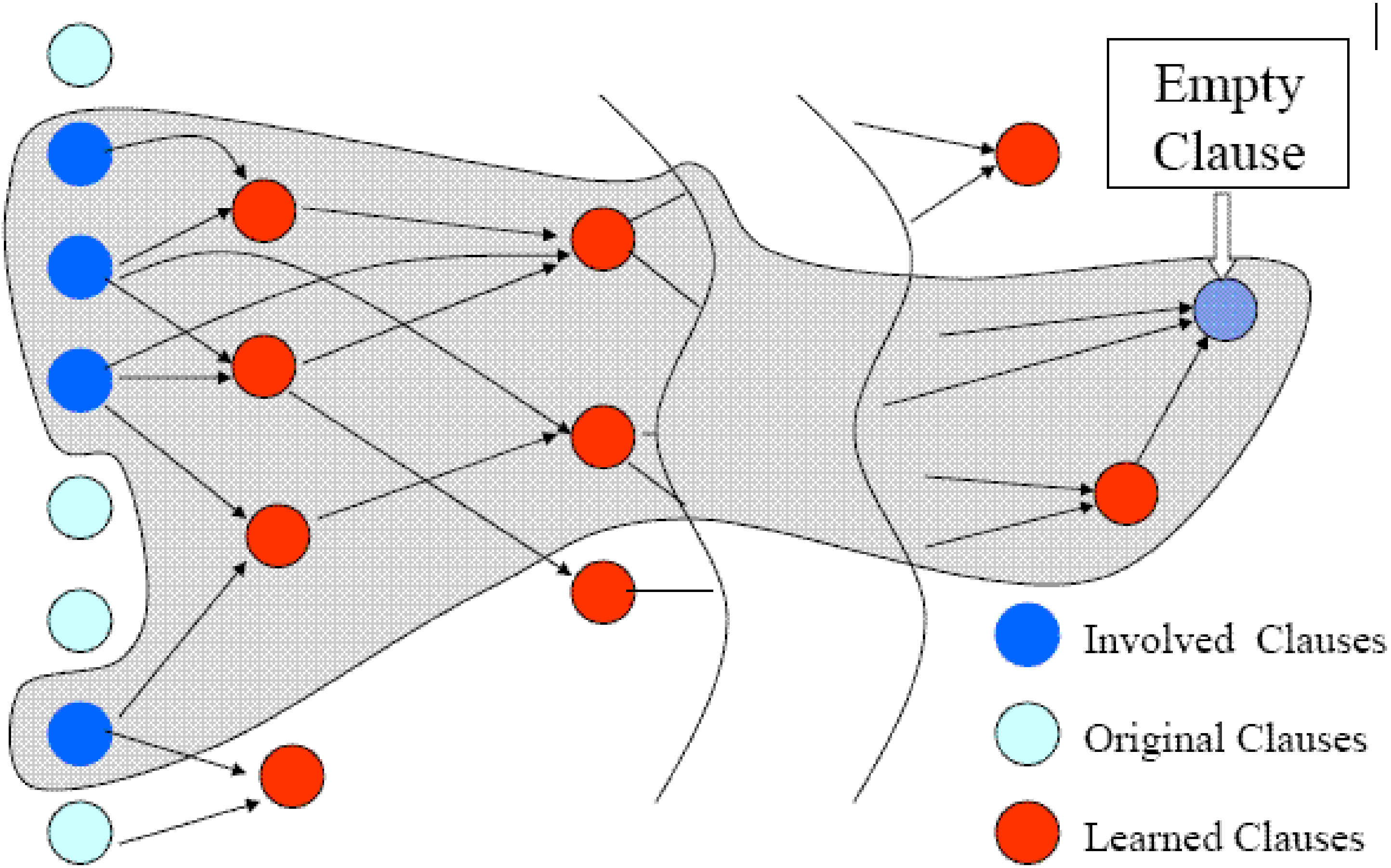
- Nodes are clauses
- Edges are applications of resolution



Proof Checker

- Given resolution graph, how to check it?
- Traverse it, checking that each node is correctly obtained from its predecessor nodes using resolution

Unsatisfiable Core



Incremental SAT Solving

- Suppose you have not just one SAT problem to solver, but many “slightly differing” problems over the same variables
- Can we re-use the search over many problems?
  - i.e. perform only “incremental” work

Operations Needed

1. Adding clauses
2. Deleting clauses

- Which is easy and which is hard?
  - If previous problem is unsat, how does an operation change it?
  - If previous is sat?