HW 6: (Un)Decidability

Assigned: March 11, 2010  Due in drop box by 5 pm: March 19, 2010

Note: Take time to write clear and concise solutions. Confused and long-winded answers may be penalized. Consult the course webpage for course policies on collaboration.

1. (6 points)
   
   Let $A$ be the following language:
   
   \[
   \{ G \mid G \text{ is a context-free grammar that generates a string in } \{0, 1\}^* \text{ which the number of 1s is a multiple of 7 and the number of 0s is a multiple of 5} \}.
   \]

   Prove that $A$ is decidable.

2. (8 points)

   Let $\Sigma = \{0, 1\}$, and let $A$ be a language over $\Sigma$. Prove that $A$ is Turing-recognizable if and only if there exists a decidable language $B$ where

   \[
   A = \{ w \mid \exists x \in \Sigma^* \text{ s.t. } (w, x) \in B \}
   \]

   Note: $B$ is like a language of “witness strings”. In the above definition, the string $x$ is like a witness to the fact that $w$ is in $A$, in the following sense: given $x$, we can decide if $w \in A$ even if that was not possible without $x$.

3. (8 points)

   Let $\text{PREF}^{\text{DFA}} = \{ \langle M \rangle \mid M \text{ is a DFA that accepts any prefix of any string that it accepts} \}$.  
   
   Prove that $\text{PREF}^{\text{DFA}}$ is decidable.

4. (8 points)

   Let $A$ be a Turing-recognizable language consisting of descriptions of Turing machines, $\{ \langle M_1 \rangle, \langle M_2 \rangle, \ldots \}$, where every $M_i$ is a decider. Prove that some decidable language $D$ is not decided by any decider $M_i$ whose description appears in $A$.
   
   (Hint: You may find it helpful to consider an enumerator for $A$.)