Introduction to Reachabilityability

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Outline

- Introduction to optimal control
- Reachability as an optimal control problem
- Various shades of reachability
Goal of This Presentation

- Reachability is nothing but an optimal control/differential game problem
- Everything in reachability ultimately amounts to solving a PDE.
- Any (small enough) optimal control problem (including reachability problems) can be solved using the Level Set Toolbox.
Optimal Control

- Optimize a cost function subject to system dynamics.

- Discrete-time systems:

\[
J^* = \min_{x(\cdot), u(\cdot)} \sum_{t=0}^{N-1} C(x(t), u(t), t) + l(x(N))
\]

subject to \( x(t + 1) = f(x(t), u(t), t) \); \( x(0), N \) – fixed

- Continuous-time systems:

\[
J^* = \min_{x(\cdot), u(\cdot)} \int_{t_0}^{T} C(x(t), u(t), t) dt + l(x(T))
\]

subject to \( \dot{x} = f(x, u, t) \); \( x(t_0), T \) – fixed
A Quick Example: Dubins Car

\[ \min_{x(\cdot), u(\cdot)} \int_0^1 \left( x^T Q x + u^T R u \right) dt + x(1)^T Q_f x(1) \]

subject to

- \[ \dot{x} = v \cos \phi \]
- \[ \dot{y} = v \sin \phi \]
- \[ \dot{\phi} = \omega \]
- \[ x(0) = [1, 1, \frac{\pi}{6}] \]

Quick question: what are we trying to do to the system here?
Solving Optimal Control Problems

- **Approach 1: Calculus of Variations (CoV)**
  - Takes an optimization perspective
  - Uses Lagrange multipliers to eliminate constraints
  - Derive first-order optimality conditions
  - Globally optimal solution is not guaranteed

\[
J^* = \min_{x(\cdot), u(\cdot)} \int_{t_0}^{T} C(x(t), u(t), t) \, dt + l(x(T))
\]

subject to \( \dot{x} = f(x, u, t); \quad x(t_0), T \text{ fixed} \)
Approach 2: Principle of Dynamic Programming

- Gives the globally optimal solution.

Principle: *The optimal state trajectory remains optimal at intermediate points in time.*
Dynamic Programming Example*

Step T: \( V_T(x = 4) = 0 \)

Step T-1: \( V_{T-1}(2) = w_{down}(2) + V_T(4) \)
\[
V_{T-1}(2) = 1 + 0 = 1
\]
\( V_{T-1}(3) = w_{right}(3) + V_T(4) \)
\[
V_{T-1}(3) = 4 + 0 = 4
\]

Step T-2:
\[
V_{T-2}(1) = \min \left[(w_{right}(1) + V_{T-1}(2)), (w_{down}(1) + V_{T-1}(3))\right]
\]
\[
V_{T-2}(1) = \min \left[(5 + 1), (1 + 4)\right] = 5
\]

*Shamelessly copied from Sylvia Herbert’s presentation
Magic of Dynamic Programming

- Compute the set of states that can reach the target set within 4s?
  - Assume that we can compute the set of states that can reach any other given set of states within 1s.
Recall our optimal control problem:

\[
\text{Minimize } \quad J(x, t) = \int_t^T C(x(t), u(t)) dt + l(x(T)) \\
\text{Subject to } \quad \dot{x} = f(x, u, t)
\]

Define cost from \((t, x)\)

\[
V(x(t), t) = \min_{u(\cdot)} \left[ \int_t^T C(x(t), u(t)) dt + l(x(T)) \right]
\]

We are interested in finding cost from state \(x\) and time 0

\[
V(x(0), 0) = \min_{u(\cdot)} \left[ \int_0^T C(x(t), u(t)) dt + l(x(T)) \right]
\]
Dynamic programming principle implies that:

\[ V(x(t), t) = \min_u \left[ V(x(t + \delta), t + \delta) + \int_t^{t+\delta} C(x, u) dt \right] \]
Principle of Dynamic Programming

\[ V(x(t), t) = \min_u \left[ V(x(t + \delta), t + \delta) + \int_t^{t+\delta} C(x, u) dt \right] \]

\[ V(x(t), t) + \frac{dV}{dt}(x(t), t)\delta + \nabla V(x(t), t) \cdot \frac{dx}{dt}\delta + C(x, u)\delta \]

Taylor Expansion

Approximation
Principle of Dynamic Programming

\[ V(x(t), t) = \min_{u(t)} \left[ V(x(t), t) + \frac{dV}{dt} (x(t), t)\delta + \nabla V(x(t), t) \cdot \frac{dx}{dt} \delta + C(x, u)\delta \right] \]

\[ V(x(t), t) = V(x(t), t) + \min_{u(t)} \left[ \frac{dV}{dt} (x(t), t)\delta + \nabla V(x(t), t) \cdot \frac{dx}{dt} \delta + C(x, u)\delta \right] \]

\[ V(x(t), t) = V(x(t), t) + \min_{u(t)} \left[ \frac{dV}{dt} (x(t), t)\delta + \nabla V(x(t), t) \cdot \frac{dx}{dt} \delta + C(x, u)\delta \right] \]

\[ 0 = \min_{u(t)} \left[ \frac{dV}{dt} (x(t), t)\delta + \nabla V(x(t), t) \cdot \frac{dx}{dt} \delta + C(x, u)\delta \right] \]

\[ 0 = \min_{u(t)} \left[ \frac{dV}{dt} (x(t), t) + \nabla V(x(t), t) \cdot \frac{dx}{dt} + C(x, u) \right] \]
Principle of Dynamic Programming

\[ 0 = \min_{u(t)} \left[ \frac{dV}{dt} (x(t), t) + \nabla V(x(t), t) \cdot \frac{dx}{dt} + C(x, u) \right] \]

\[ \frac{dV}{dt} + \min_u \{\nabla V(x(t), t) \cdot f(x, u) + C(x, u)\} = 0 \]

\[ V(x(T), T) = l(x(T)) \]

Hamilton-Jacobi Bellman PDE

Final Value of PDE
Hamilton-Jacobi Bellman (HJB) PDE

Problem:

Minimize \( J(x, t) = \int_t^T C(x(t), u(t)) dt + l(x(T)) \)
Subject to \( \dot{x} = f(x, u, t) \)

Solution

A final-value PDE

\[
\frac{dV}{dt} + \min_{u} \{ \nabla V(x(t), t) \cdot f(x, u) + C(x, u) \} = 0
\]

\[
V(x(T), T) = l(x(T))
\]

- What is \( V(x(t), t) \)?
- Where is my optimal control?
Curse of Dimensionality

- Wow! One single PDE for any optimal control problem.
  - What is the problem here?

![Graph showing computation time vs number of dimensions: 1D: 0.2 s, 2D: 0.5 s, 3D: 10 s, 4D: 1525 s.](image)
Level Set Toolbox

- Developed by Prof. Ian Mitchell during his PhD
- Can solve any initial-value PDE of the form:

\[
\frac{dV}{dt} + H^* (x, \nabla V(x(t), t), t) = 0
\]
\[
V(x(0), 0) = l(x(0))
\]

But optimal control problem is a final value PDE .... :

\[
\frac{dV}{dt} + \min_u \{\nabla V(x(t), t) \cdot f(x, u) + C(x, u)\} = 0
\]
\[
V(x(T), T) = l(x(T))
\]
Good news: They are interchangeable!

\[ \frac{dV}{dt} + H^* (x, \nabla V(x(t), t), t) = 0 \]

\[ V(x(0), 0) = l(x(0)) \]

\[ W(x, T - t) = V(x, t) \]

\[ \frac{dW}{dt} - H^* (x, \nabla W(x(t), t), t) = 0 \]

\[ W(x(T), T) = l(x(T)) \]

So we can solve any optimal control problem with the toolbox!
Optimal Control: Quick Recap

- **Optimal control problem:**
  - Optimize a cost function subject to system dynamics.

- **Two approaches:**
  - **Calculus of Variations:**
    - Gives local solutions, but faster to compute.
  - **Dynamic Programming:**
    - Gives global solution
    - A Final-value PDE needs to be solved
      - May not even have a classical solution!
      - Curse of dimensionality!
      - Can be solved using Level Set Toolbox for low dimensional problems.
**Reachability**

- **Problem:** Find the set of all states that can reach a given set of states \( \mathcal{L} \) within a time duration of \( T \).

\[
\mathcal{R}(T) = \{ x_0 : \exists u, \text{s.t. } x(\cdot) \text{satisfies } \dot{x} = f(x, u), x(0) = x_0; \exists t \in [0, T], \text{s.t. } x(t) \in \mathcal{L} \}
\]

- **Any thoughts?**
Reachability

- **Hint 1:** Define a function $l(x)$ such that,

$$\mathcal{L} = \{x: l(x) \leq 0\}$$

Now consider the problem,

$$V(x(t), t) = \min_u l(x(T))$$

Subject to $\dot{x} = f(x, u, t)$

- What does $V(x(t), t)$ represent?
What are $V(x(t), t)$ for each of the following trajectories?

$$V(x(t), t) = \min_u l(x(T))$$
Reachability

- So what does $V(x(t), t)$ represent?
  - The value of $l(x)$ that we will reach at time $T$

- How do I answer my original question?

$$x(0) \in \mathcal{R}(T) \Leftrightarrow V(x(0), 0) \leq 0$$

$$x(0) \notin \mathcal{R}(T) \Leftrightarrow V(x(0), 0) > 0$$
Reachability

- So reachability is nothing but an optimal control problem.

\[
\min_u l(x(T))
\]

Subject to \( \dot{x} = f(x, u, t) \)

\[ \mathcal{L} = \{ x : l(x) \leq 0 \} \]

- What is the corresponding PDE?

\[
\frac{dV}{dt} + H^* (x, \nabla V(x(t), t), t) = 0
\]

\[ V(x(T), T) = l(x(T)) \]

\[ H^* = \min_u \{ \nabla V(x(t), t) \cdot f(x, u, t) \} \]

- How to get "target-reaching" control?

\[ u^* = \arg\min_u \{ \nabla V(x(t), t) \cdot f(x, u, t) \} \]
What are $V(x(t), t)$ for each of the following trajectories?

$\mathcal{R}(T) = \{x_0: \exists u, \text{s.t. } x(\cdot) \text{ satisfies } \dot{x} = f(x, u), x(0) = x_0; \exists t \in [0, T], \text{s.t. } x(t) \in \mathcal{L}\}$

$x(0) \in \mathcal{R}(T) \iff V(x(0), 0) \leq 0$
What’s going on?

- Need to account for the fact that trajectories can reach the target but then leave it.

\[ \min_u \left( \min_{t \in [0, T]} l(x(t)) \right) \]

Subject to \( \dot{x} = f(x, u, t) \)

\[ \mathcal{L} = \{ x : l(x) \leq 0 \} \]

- Does this fix the issue?

\[ l(x) = 1 \]

\[ V(x(t), t) = \min_u l(x(T)) \]
Freezing the Trajectories in the Target

- Need to account for the fact that trajectories can reach the target but then escape it.

$$\min_u \left( \min_{t \in [0,T]} l(x(t)) \right)$$

Subject to $\dot{x} = f(x, u, t)$

$L = \{x: l(x) \leq 0\}$

- What is the corresponding PDE?

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t), t), t)\} = 0$$

$V(x(T), T) = l(x(T))$

$H^* = \min_u \{\nabla V(x(t), t) \cdot f(x, u, t)\}$
Reachability: Reachable Sets vs Tubes

- **Backward Reachable Set (BRS):** the set of all states that can reach a target set of states $\mathcal{L}$ exactly at time $T$.

  $$\mathcal{R}'(T) = \{x_0: \exists u, \text{s.t. } x(\cdot) \text{ satisfies } \dot{x} = f(x, u), x(0) = x_0; x(T) \in \mathcal{L}\}$$

  $$\begin{align*}
  &\min_u l(x(T)) \\
  \text{Subject to } &\dot{x} = f(x, u, t) \\
  \mathcal{L} &= \{x: l(x) \leq 0\}
\end{align*}$$

  $$\begin{align*}
  &\frac{dV}{dt} + H^* (x, \nabla V(x(t), t), t) = 0 \\
  V(x(T), T) &= l(x(T)) \\
  H^* &= \min_u \{\nabla V(x(t), t) \cdot f(x, u, t)\}
\end{align*}$$

- **Backward Reachable Tube (BRT):** the set of all states that can reach a target set of states $\mathcal{L}$ within a duration of time $T$.

  $$\mathcal{R}(T) = \{x_0: \exists u, \text{s.t. } x(\cdot) \text{ satisfies } \dot{x} = f(x, u), x(0) = x_0; \exists t \in [0, T], \text{s.t. } x(t) \in \mathcal{L}\}$$

  $$\begin{align*}
  &\min_u \left( \min_{t \in [0, T]} l(x(t)) \right) \\
  \text{Subject to } &\dot{x} = f(x, u, t) \\
  \mathcal{L} &= \{x: l(x) \leq 0\}
\end{align*}$$

  $$\begin{align*}
  &\frac{dV}{dt} + \min\{0, H^* (x, \nabla V(x(t), t), t)\} = 0 \\
  V(x(T), T) &= l(x(T)) \\
  H^* &= \min_u \{\nabla V(x(t), t) \cdot f(x, u, t)\}
\end{align*}$$
Backward Reachable Tube: Example

Target set

Backward reachable tubes

$t = -1$

$t = -2$

$t = -3$

$t = -4$
Backward Reachable Set: Example

Backward reachable sets

- $t = -1$
- $t = -2$
- $t = -3$
- $t = -4$

Target set
Computing Backward Reachable Tube

1. Define target set $\mathcal{L}$ for the system to reach within a given time horizon:

2. Define implicit level set function for final time $l(z)$, $\mathcal{L} = \{z: l(z) \leq 0\}$

3. Find an appropriate value function $V(z(t), t)$

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t), t), t)\} = 0$$

$$V(x(T), T) = l(x(T))$$

$$H^* = \min_u \{\nabla V(x(t), t) \cdot f(x, u, t)\}$$

4. Retrieve zero sub-level of level set function at initial time
Reachability: Key Takeaways

- Reachability is just an optimal control problem.
- Backward Reachable Set (BRS) vs Backward Reachable Tube (BRT)
- Both can be computed using the Level Set Toolbox.
- Suffers from the curse of dimensionality
Introducing the Disturbance

- Suppose our dynamics were: \( \dot{x} = f(x, u, d) \)
  - \( u \) – control, \( d \) – disturbance

- And cost were: \( J(x, t) = \int_t^T C(x(t), u(t), d(t))dt + l(x(T)) \)

- Now, we want to solve the following \textit{differential} game:

\[
V(x(t), t) = \min_{u(\cdot)} \max_{d(\cdot)} \left[ \int_t^T C(x(t), u(t), d(t))dt + l(x(T)) \right]
\]

- A similar PDE can be derived in this case (called HJI PDE)
<table>
<thead>
<tr>
<th></th>
<th>Static</th>
<th>Evolving Over Time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>One agent</strong></td>
<td>Optimization</td>
<td>Optimal Control</td>
</tr>
<tr>
<td>(input)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Multiple agents</strong></td>
<td>Game Theory</td>
<td>Differential Games</td>
</tr>
<tr>
<td>(input and disturbance)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Hamilton-Jacobi Isaacs (HJI) PDE

Problem:

Minimize \[ J(x, t) = \int_t^T C(x(t), u(t), d(t))dt + l(x(T)) \]
Subject to \[ \dot{x} = f(x, u, d, t) \]

Solution

A final-value PDE

\[ \frac{dV}{dt} + \min_u \max_d \{\nabla V(x(t), t) \cdot f(x, u, d) + C(x, u, d)\} = 0 \]
\[ V(x(T), T) = l(x(T)) \]

- What is \( V(x(t), t) \)?
- Where is my optimal control?
Reachability With Disturbance

- **Problem**: Find the set of all states that can reach a given set of states $\mathcal{L}$ despite the disturbance within a time duration of $T$.

$$\mathcal{R}(T) = \{x_0 : \exists u, \text{s.t. } \forall d, x(\cdot) \text{ satisfies } \dot{x} = f(x, u, d), x(0) = x_0; \exists t \in [0, T], \text{s.t. } x(t) \in \mathcal{L}\}$$

- What does this definition mean?
- How to solve this problem?
Reachability With Disturbance

- We can again formulate it as a differential game

\[
\min_u \max_d \min_{t \in [0,T]} l(x(t))
\]
Subject to \( \dot{x} = f(x, u, d, t) \)
\( \mathcal{L} = \{ x : l(x) \leq 0 \} \)

- What is the corresponding PDE?

\[
\frac{dV}{dt} + \min \{ 0, H^*(x, \nabla V(x(t), t), t) \} = 0
\]
\( V(x(T), T) = l(x(T)) \)
\( H^* = \min_u \max_d \{ \nabla V(x(t), t) \cdot f(x, u, d, t) \} \)

- How to get "target-reaching" control?

\[
\mathbf{u}^* = \arg \min_u \max_d \{ \nabla V(x(t), t) \cdot f(x, u, d, t) \}
\]
Reachability With Disturbance: Key Takeaways

- Very similar to classic reachability; just an extra max
- Backward Reachable Set (BRS) vs Backward Reachable Tube (BRT)
- Both can be computed using the Level Set Toolbox.
- Suffers from the curse of dimensionality
Reachability With Disturbance: Quick Trivia

**Problem:** Find the set of all states that can reach a given set of states $\mathcal{L}$ for some disturbance within a time duration of $T$.

$$\mathcal{R}(T) = \{x_0 : \exists u, \exists d, \text{s.t. } x(\cdot) \text{ satisfies } \dot{x} = f(x, u, d), x(0) = x_0; \exists t \in [0, T], \text{s.t. } x(t) \in \mathcal{L}\}$$

- What does this definition mean?
- How to solve this problem?
Reachability With Disturbance: Trivia Solution

- Disturbance is like control here

\[
\min_u \min_d \min_{t \in [0,T]} l(x(t))
\]
Subject to \( \dot{x} = f(x, u, d, t) \)
\( \mathcal{L} = \{x: l(x) \leq 0\} \)

- What is the corresponding PDE?

\[
\frac{dV}{dt} + \min\{0, H^* (x, \nabla V(x(t), t), t)\} = 0
\]

\[
V(x(T'), T') = l(x(T'))
\]

\[
H^* = \min_u \min_d \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}
\]
Shades of Reachability: Avoid Set

- **Problem:** Find the set of all states that can *avoid* a given set of states $\mathcal{L}$ *despite the disturbance* for a time duration of $T$.

\[
\mathcal{R}(T) = \{ x_0 : \exists d, \text{ s.t. } \forall u, x(\cdot) \text{ satisfies } \dot{x} = f(x, u, d), x(0) = x_0; \exists t \in [0, T], \text{ s.t. } x(t) \in \mathcal{L} \}
\]

- What does this definition mean?
- How to solve this problem?
Reachability: Avoid Set

- Simply exchange the role of input and disturbance
  \[
  \max_{u} \min_{d} \min_{t \in [0,T]} l(x(t))
  \]
  Subject to \( \dot{x} = f(x, u, d, t) \)
  \( \mathcal{L} = \{x : l(x) \leq 0\} \)

- What is the corresponding PDE?
  \[
  \frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t), t), t)\} = 0
  \]
  \[
  V(x(T), T) = l(x(T))
  \]
  \[
  H^* = \max_{u} \min_{d} \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}
  \]

- How to get "target-avoiding" control?
  \[
  u^* = \arg\max_{u} \min_{d} \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}
  \]
Avoid Set: Example

- What do the green sets represent here?
Problem: Find the set of all states that I can reach from a given set of states $\mathcal{L}$ at time $T$.

Initial Set ($\mathcal{L}$)

Forward Reachable Set $\mathcal{R}(T)$

$F(T) = \{ x_T : \exists u, s.t. x(\cdot) satisfies \dot{x} = f(x, u), x(0) \in \mathcal{L}; x(T) = x_T \}$

What does this definition mean?

How to solve this problem?
Define a function $l(x)$ such that,
\[
L = \{x : l(x) \leq 0\}
\]

Now consider the problem,
\[
V(x(t), t) = \max_u l(x(T)) \\
\text{Subject to } \dot{x} = f(x, u, t)
\]

Will this work?
Again it is an optimal control problem.

\[
\max_u l(x(T))
\]
Subject to \( \dot{x} = f(x, u, t) \)
\( \mathcal{L} = \{x: l(x) \leq 0\} \)

What is the corresponding PDE?

\[
\frac{dV}{dt} + H^* (x, \nabla V(x(t), t), t) = 0
\]
\[ V(x(0), 0) = l(x(0)) \]
\[ H^* = \max_u \{\nabla V(x(t), t) \cdot f(x, u, t)\} \]
Forward Reachable Set Trivia

- **Problem:** Find the set of all states that I can reach *from* a given set of states $\mathcal{L}$ *despite the disturbance* at time $T$.

  \[ F(T) = \{ x_T : \exists u, \text{s.t.} \forall d, x(\cdot) \text{ satisfies } \dot{x} = f(x, u, d), x(0) \in \mathcal{L}; \ x(T) = x_T \} \]

- **Solve the following PDE:**

  \[
  \frac{dV}{dt} + H^*(x, \nabla V(x(t), t), t) = 0 \\
  V(x(0), 0) = l(x(0)) \\
  H^* = \max_u \min_d \{ \nabla V(x(t), t) \cdot f(x, u, t) \} 
  \]
Forward Reachable Set: Key Takeaways

- Gives an initial value PDE
- Forward Reachable Set (FRS) vs Forward Reachable Tube (FRT)
- Both can be computed using the Level Set Toolbox.
- Suffers from the curse of dimensionality
Shades of Reachability: Obstacles

- **Problem:** Find the set of all states that can *reach* a given set of states $\mathcal{L}$ *without hitting the obstacle* $(G)$ *despite the disturbance* within a time duration of $T$.

\[ \mathcal{R}(T) = \left\{ x_0 : \exists u, \text{s.t. } \forall d, x(\cdot) \text{satisfies } \dot{x} = f(x, u, d), x(0) = x_0; \right\} \]

\[ \forall t \in [0, T], x(t) \notin G \text{ and } \exists t \in [0, T], \text{s.t. } x(t) \in \mathcal{L} \]

- What does this definition mean?
Reachability With Obstacles

- Again can be formulated as a differential game

- Value function can be shown to satisfy the following equation:

\[
\max \left\{ \frac{dv}{dt} + \min \{0, H \ast (x, \nabla V(x(t), t), t)\}, g(x) - V(x, t) \right\} = 0
\]

\[
V(x(T), T) = \max\{l(x, T), g(x, T)\}
\]

\[
H^* = \min \max_u \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}
\]

- How to get "target-reaching" control without hitting the obstacle?

\[
u^* = \arg\min_u \max_d \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}
\]
Reachability With Obstacles: Example
Shades of Reachability: Key Takeaways

- Everything in reachability ultimately amounts to solving a PDE.
- Different min-max combinations appear in the PDE based on what control and disturbance are trying to do.
- Min with zero appears in the PDE depending on whether a set or a tube is being computed.
- Initial or final value PDE is solved based on whether a FRS or a BRS is being computed.
- Obstacles can be considered.
- Any combination above can be computed using the Level Set Toolbox.
Reachability: Final Remarks

- Reachability theory has much more rigorous mathematical foundation.

- Time-varying targets/obstacles can also be treated very easily.
Thank You!

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