EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 12 – Design via State Space

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Lecture abstract

Topics covered in this presentation

- Controller design
- Controllability
- Observer design
- Observability

Chapter outline

Intro, [1, p. 664]

Concept

- Design via state space
  - Can be applied to a wider class of systems than transform methods
    - Systems with nonlinearities
    - Multiple-input, multiple output (MIMO) systems
    - Systems of higher order than 2
  - We will focus on the application to linear systems
  - Specify all CL poles
    - Parameters for each CL pole
    - Technique for finding these parameter values
  - Cannot specify CL zero locations
  - Sensitive to parameter changes
  - Wide range of computational support
    - Loss of graphic insight into a design problem

State-variable FB control, [1, p. 665]

Concept

- n^{th}-order CL characteristic equation (CE)

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = 0$$

- There are n coefficients whose values determine the n CL poles
- Introduce n adjustable parameters into the system and relate them to the n coefficients, so that we can place the n CL poles

State-variable FB control, [1, p. 666]

Concept

- Before, output-variable FB, now, state-variable FB
  - Each state variable is fed back to the control, u, through a gain, k_i
  - *State-variable FB gain:* $-K$
- CL system is plant with state-variable FB

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= Ax + B(-Kx + r) \\ &= (A - BK)x + Br \\ y &= Cx \end{aligned}$$

State-variable FB control, [1, p. 666]

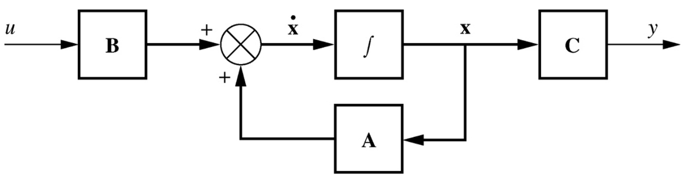

Figure: Plant with state-variable FB

State-variable FB control, [1, p. 666]

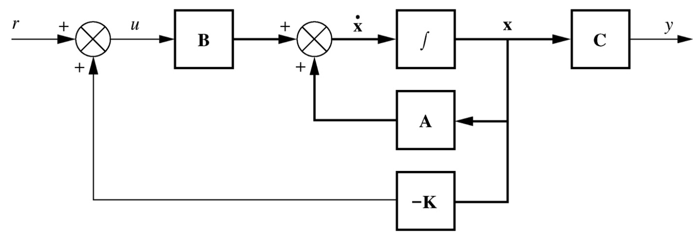

Figure: State-space representation of a plant

Methodology, [1, p. 668]

Procedure

- Pole placement for plants in phase-variable (PV) form

1. Represent the plant in PV form
2. FB each PV to the input of the plant through a gain, k_i
3. Find the CE for the CL system
4. Decide upon all CL pole locations and determine equivalent CE
5. Equate like coefficients of the CE and solve for k_i

State-variable FB control in PV form, [1, p. 668]

Concept

- Plant

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

- Plant CE

$$\det(sI - A) = $$
$$s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0$$

State-variable FB control in PV form, [1, p. 668]

Concept

- State-variable FB

$$u = -Kx; \quad K = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$

- CL system

$$A - BK =$$
$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \dots & -(a_{n-1} + K_n) \end{bmatrix}$$

- CL system CE

$$\det(sI - (A - BK)) =$$
$$s^n + (a_{n-1} + k_n)s^{n-1} + \dots + (a_1 + k_2)s + (a_0 + k_1) = 0$$

State-variable FB control in PV form, [1, p. 669]

Concept

- Desired CL system CE

$$\det(sI - (A - BK)) =$$
$$s^n + d_{n-1}s^{n-1} + \dots + d_1 s + d_0 = 0$$

$$d_i = a_i + k_{i+1}; \quad i = 0, 1, 2, \dots, n-1$$

- CL system TF
  - Denominator polynomial: the CE
  - Numerator polynomial: formed from the coefficients of the output coupling matrix, C, for systems written in PV form
    - Same for plant and CL system

Example, [1, p. 669]

Example (Controller design for PV form)

- *Problem:* Design the PV FB gains to yield
  - $\%OS = 9.5\%$
  - $T_s = 0.74$ seconds
- *Solution:* On the board

Definitions, [1, p. 672]

- *Controllability:* If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise the system is uncontrollable.
  - Control variable, u, can be used to control the behavior of each state variable in x
  - Poles of the control system can be placed where we desire
  - Determine, a priori, whether pole placement is a viable design technique for a controller

Controllability by inspection, [1, p. 673]

Concept

- When the system matrix, A, is in diagonal or parallel form, it is apparent whether or not the system is controllable
  - A system with distinct (no repeat) eigenvalues and a diagonal system matrix, A, is controllable if the input coupling matrix, B, does not have any rows that are zero

The controllability matrix, [1, p. 674]

- In other forms, the existence of paths from the input to the state variables is not a criterion for controllability since the equations are not decoupled
  - n^{th}-order plant

$$\dot{x} = Ax + Bu$$

is completely controllable if the matrix

$$C_M = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

is of rank n, where C_M is called the controllability matrix

Example, [1, p. 675]

Example (Controllability via the controllability matrix)

- *Problem:* Determine if the system is controllable

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- *Solution:* On the board

Alternative approaches to controller design, [1, p. 676]

Concept

- For systems not represented in PV form
- 2 approaches
  - *Matching coefficients:* Matching the coefficients of

$$\det(sI - (A - BK))$$

with coefficients of the desired CE
    - Same method used for systems in PV representation
  - *Transformation:* Transforming the system to PV form, designing the control FB gain, & transforming the designed system back to its original state-variable representation

Approach – matching coefficients, [1, p. 677]

Concept

- Matching the coefficients of

$$\det(sI - (A - BK))$$

with coefficients of the desired CE
  - Leads to difficult calculations of the control gains, especially for higher-order systems not represented with PVs

Example, [1, p. 677]

Example (Controller design by matching coefficients)

- *Problem:* Design state-variable control FB gain for the plant to yield
  - $\%OS = 15\%$
  - $T_s = 0.5$ second

$$G(s) = \frac{10}{(s+1)(s+2)}$$

- *Solution:* On the board

Approach – transformation, [1, p. 678]

Procedure

1. Transform the system to PV representation
2. Design the state-variable control FB gain
3. Transform the system in PV representation back to the original representation

Approach – transformation, [1, p. 678]

Procedure

1. Transform the system to PV representation
   - Plant not in PV representation

$$\dot{z} = Az + Bu$$
$$y = Cz$$

with controllability matrix

$$C_{M_z} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

Approach – transformation, [1, p. 678]

Procedure

1. Transform the system to PV representation
   - Assume the system can be transformed into the PV representation with the transformation

$$z = Px$$

Transformed plant

$$\dot{x} = P^{-1}APx + P^{-1}Bu$$
$$y = CPx$$

with controllability matrix

$$C_{M_x} = P^{-1}C_{M_z}$$

   - Solving for P

$$P = C_{M_z}C_{M_x}^{-1}$$

Approach – transformation, [1, p. 678]

Procedure

2. Design the control FB
   - Include both state-variable control FB and input

$$u = -K_x x + r$$

   - Transformed plant with state-variable control FB

$$\dot{x} = (P^{-1}AP - P^{-1}BK_x)x + P^{-1}Br$$
$$y = CPx$$

   - Zeros of this CL system are determined from the polynomial formed from the elements of CP

Approach – transformation, [1, p. 679]

Procedure

3. Transform the system in PV representation back to the original representation
   - Plant not in PV representation with state-variable control FB

$$\dot{z} = (A - BK_xP^{-1})z + Br$$
$$y = Cz$$

   - State-variable control FB gain

$$K_z = K_xP^{-1}$$

   - Zeros of the CL TF are the same as the zeros of the uncompensated plant

Example, [1, p. 679]

Example (Controller design by transformation)

- *Problem:* Design state-variable control FB gain for the plant to yield
  - $\%OS = 20.8\%$
  - $T_s = 4$ seconds

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+5)}$$

- *Solution:* On the board

- **12 Design via state space**
  - 12.1 Introduction
  - 12.2 Controller design
  - 12.3 Controllability
  - 12.4 Alternative approaches to controller design
  - **12.5 Observer design**
  - 12.6 Observability
  - 12.7 Alternative approaches to observer design
  - 12.8 Steady-state error design via integral control

Observer motivation, [1, p. 682]

Concept

- Controller design relies upon access to the state variables for FB through adjustable gains
- Estimate states can be fed to the controller
- *Observer:* Estimator used to calculate state variables that are not accessible from the plant

- Plant

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- Observer

$$\dot{\hat{x}} = A\hat{x} + Bu$$
$$\hat{y} = C\hat{x}$$

- Observer error

$$\dot{e_x} = \dot{x} - \dot{\hat{x}} \qquad = A(x - \hat{x})$$
$$y - \hat{y} = C(x - \hat{x}) \qquad = Ce_x$$

Observer motivation, [1, p. 683]

Concept

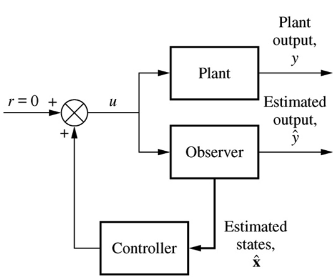

Figure: State-FB design using an OL observer

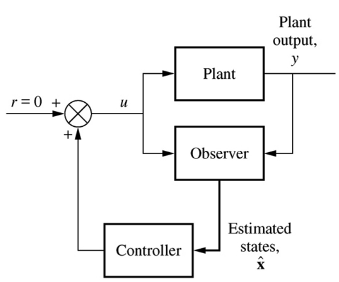

Figure: State-FB design using a CL observer

Observer motivation, [1, p. 683]

Concept

- Dynamics of the difference between the actual & estimated states is unforced, & if the plant is stable, this difference, due to the differences in initial state vectors, $\to 0$
- Speed of convergence between the actual state & the estimated state is the same as the TR of the plant since the CE of the observer error is the same as the plant
- Convergence is too slow, $\to$ speed up the observer and make its response time much faster than that of the controlled CL system, $\to$ the controller will receive the estimated states instantaneously
  - Error between the outputs of the plant and the observer is fed back to the derivatives of the observer's states
  - The system corrects to drive this error $\to 0$
  - Design a desired TR into the observer that is much quicker than that of the plant or controlled CL system

Observer motivation, [1, p. 683]

Concept

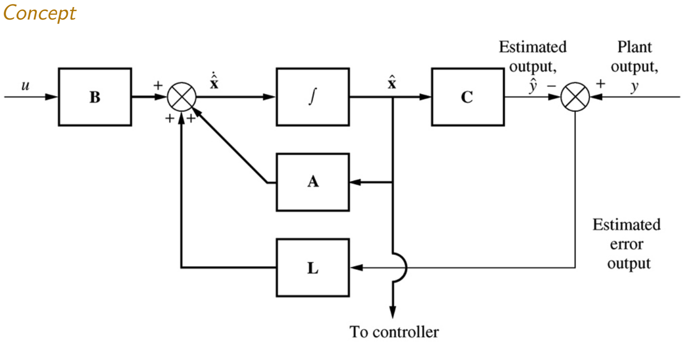

Figure: State-FB design using a CL observer exploded view showing FB arrangement to reduce state-variable estimation error

Observer motivation, [1, p. 683]

Concept

- Observer canonical form yields an easy solution for the observer FB gain
- *Observer FB gain, L:* Ensures the TR of the observer is faster than the response of the controlled loop in order to yield a rapidly updated estimate of the state vector

Design methodology, [1, p. 684]

Procedure

1. Find error system, i.e., state equations for error between actual state vector & estimated state vector, $x - \hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

2. Find CE for error system
3. Evaluate required observer FB gain, L, to meet rapid TR for observer
4. Select eigenvalues of observer to yield stability & desired TR that is faster than controlled CL response

Definitions, [1, p. 690]

- *Observability:* If the initial-state vector, $x(t_0)$, can be found from inputs, $u(t)$, and measurements, $y(t)$, over a finite interval of time from t_0, the system is said to be observable; otherwise the system is said to be unobservable.
  - Knowledge of measured output variables, y, and control inputs, $u(t)$, can be used to observe the behavior of each state variable in x
  - Poles of the observer system can be placed where we desire
  - Determine, a priori, whether pole placement is a viable design technique for an observer

Observability by inspection, [1, p. 690]

Concept

- When the system matrix, A, is in diagonal or parallel form, it is apparent whether or not the system is observable
  - A system with distinct (no repeat) eigenvalues and a diagonal system matrix, A, is observable if the output coupling matrix, C, does not have any columns that are zero

The observability matrix, [1, p. 691]

- In other forms, the existence of paths from the output to the state variables is not a criterion for observability since the equations are not decoupled
  - n^{th}-order plant

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is completely observable if the matrix

$$O_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is of rank n, where O_M is called the observability matrix

Example, [1, p. 691]

Example (Observability via the observability matrix)

- *Problem:* Determine if the system is observable

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}$$

- *Solution:* On the board

Example, [1, p. 692]

Example (Unobservability via the observability matrix)

- *Problem:* Determine if the system is observable

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{21}{4} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} 5 & 4 \end{bmatrix}$$

- *Solution:* On the board

Alternative approaches to controller design, [1, p. 676]

Concept

- For systems not represented in observer canonical form
- 2 approaches
  - *Matching coefficients:* Matching the coefficients of

    $$\det(sI - (A - LC))$$

    with coefficients of the desired CE
    - Same method used for systems in PV representation
  - *Transformation:* Transforming the system to observer canonical form, designing the observer FB gain, & transforming the designed system back to its original state-variable representation

Approach - matching coefficients, [1, p. 693]

Concept

- Matching the coefficients of

  $$\det(sI - (A - LC))$$

  with coefficients of the desired CE
  - Leads to difficult calculations of the observer FB gain, especially for higher-order systems not in PV representation

Example, [1, p. 698]

Example (Observer design by matching coefficients)

- *Problem:* Design an observer FB gain for the system in PV representation with a TR described by
  - $\zeta = 0.7$
  - $\omega_n = 100$

$$G(s) = \frac{407(s + 0.916)}{(s + 1.27)(s + 2.69)}$$

- *Solution:* On the board

Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation
2. Design the observer FB gain
3. Transform the system in PV representation back to the original representation

Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation
   - Plant not in PV representation

$$\dot{z} = Az + Bu$$
$$y = Cz$$

with observability matrix

$$O_{M_z} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation
   - Assume the system can be transformed into the PV, x, representation with the transformation

$$z = Px$$

Transformed plant

$$\dot{x} = P^{-1}APx + P^{-1}Bu$$
$$y = CPx$$

with observability matrix

$$O_{M_x} = O_{M_z}P$$

   - Solving for P

$$P = O_{M_z}^{-1}O_{M_x}$$

Approach – transformation, [1, p. 695]

Procedure

2. Design the observer FB
   - Transformed plant with FB gains

$$\dot{e}_x = (P^{-1}AP - L_xCP)e_x$$
$$y - \hat{y} = CPe_x$$

Approach – transformation, [1, p. 695]

Procedure

3. Transform the system in PV representation back to the original representation
   - Plant not in PV representation with observer FB gain

$$\dot{e}_z = (A - PL_xC)e_z$$
$$y - \hat{y} = Ce_z$$

   - Observer FB gain

$$L_z = PL_x$$

Example, [1, p. 695]

Example (Observer design by transformation)

- *Problem:* Design an observer in cascade form

$$G(s) = \frac{1}{(s+1)(s+2)(s+5)}$$

- *Solution:* On the board

Concepts, [1, p. 700]

Concept

- Design systems in state-space representation for steady-state error
- Error is fed forward to the controlled plant via an integrator
  - Additional state variable

$$\dot{x}_N = r - Cx$$

  - Plant

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{x}_N &= -Cx + r \\ y &= Cx \end{aligned}$$

  - Control FB

$$\begin{aligned} u &= -Kx + K_e x_N \\ &= -\begin{bmatrix} K & -K_e \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} \end{aligned}$$

Concepts, [1, p. 701]

Concept

- Error is fed forward to the controlled plant via an integrator
  - Augmented representation

$$\begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A - BK & BK_e \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix}$$

- K and K_e can be selected to yield the desired TR
- We have an additional pole to place
- Keep an eye on the CL zeros and their effect on TR

Example, [1, p. 701]

Example (Design of integral control)

- *Problem:*
  a. Design a controller without integral control to yield
     - $\%OS = 10\%$
     - $T_s = 0.5$ second

     Evaluate the steady-state error for a unit step
  b. Repeat the design using integral control. Evaluate the steady-state error for a unit step input.
- *Solution:* On the board

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.