Cuts, Trees and $\ell_1$-Embeddings of Graphs

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Abstract

Motivated by many recent algorithmic applications, this paper aims to promote a systematic study of the relationship between the topology of a graph and the metric distortion incurred when the graph is embedded into $\ell_1$ space. The main results are:

1. Explicit constant-distortion embeddings of all series-parallel graphs, and all graphs with bounded Euler number. These are the first natural families known to have constant distortion (strictly greater than 1). Using the above embeddings, algorithms are obtained which approximate the sparsest cut in such graphs to within a constant factor.

2. A constant-distortion embedding of outerplanar graphs into the restricted class of $\ell_1$-metrics known as “dominating tree metrics”. A lower bound of $\Omega(|\log n|$) on the distortion for embeddings of series-parallel graphs into (distributions over) dominating tree metrics is also presented. This shows, surprisingly, that such metrics approximate distances very poorly even for families of graphs with low treewidth, and excludes the possibility of using them to explore the finer structure of $\ell_1$-embeddability.

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1 Introduction

Let $G = (V, E)$ be an undirected graph. Each assignment of non-negative weights to the edges of $G$ naturally defines a metric space $(V, \mu)$,\footnote{More correctly, a semi-metric space since we allow $\mu(x, y) = 0$ even when $x \neq y$.} where for each pair of vertices $x, y \in V$, $\mu(x, y) = d_G(x, y)$ is the shortest-path distance between them. We say that the metric $\mu$ is supported on (or generated by) $G$. Let $(S, \rho)$ be another metric space. An embedding of $G$ into $(S, \rho)$ is a mapping $\phi : V \to S$. The distortion of $\phi$ is the smallest value $c \geq 1$ such that

$$
d_G(x, y) \leq \rho(\phi(x), \phi(y)) \leq c d_G(x, y) \quad \forall x, y \in V.
$$

Thus the distortion measures the maximum factor by which any distance is stretched in the embedding. (This is a slightly restricted definition, in which we assume that no distances are shrunk. See Section 2 for a general definition.)

In recent years, the idea of embedding a graph into a “nice” metric space with low distortion has emerged as a useful ingredient in the design and analysis of algorithms in a variety of domains. “Nice” metric spaces are those with well-studied structural properties, such as Euclidean or $\ell_1$ space, or weighted trees and distributions over them. A very incomplete list of applications includes approximation algorithms for graph and network problems, such as sparsest cut [26, 2], minimum bandwidth [17, 8], low-diameter decompositions [26], and optimal group Steiner trees [19, 10], and online algorithms for metrical task systems and file migration problems [4, 6]. These applications, together with its intrinsic mathematical interest, have made the study of low-distortion embeddings a significant field in its own right.

Most of the embeddings considered in the literature, notably [9, 4, 26], have been for metrics supported on general graphs, and give results that bound the worst-case distortion over all graphs. However, when the input graph has some special structure, it is plausible that better embeddings can be found. This is quite intuitive: it is clear that any metric is generated by the complete graph on its points, while only a very limited set of metrics can be generated by weighting the edges of, say, a tree. Thus the complexity of a metric generated by a graph $G$ intrinsically depends on the topology of $G$. At present, very little is known about this interplay between the topological and metrical properties of the graph; the search for connections between the two is emerging as an intriguing and challenging area. This paper focuses in particular on the relationship between the topology of graphs and their optimal (or near-optimal) embeddings into $\ell_1$ (i.e., real space of arbitrary dimension endowed with the $\ell_1$ metric).

Embeddings into $\ell_1$ have been widely studied, and are of special importance due to their intimate connection with the problem of finding a sparsest cut in multicommodity flow networks, which in turn is a key ingredient in approximate solutions of many other problems in such areas as VLSI layout, network routing and efficient simulations of one network by another (see, e.g., [7, 25, 23]). Although finding the exact sparsest cut is a computationally hard problem, efficient approximation algorithms for it can be obtained by embedding a natural metric associated with the optimal multicommodity flow into $\ell_1$; the approximation ratio depends essentially on the distortion.

One motivation behind this paper is the intriguing conjecture that any metric supported on a planar graph (henceforth called a planar metric) can be embedded into $\ell_1$ with constant distortion. More
generally, we conjecture that this holds for any family of graphs which excludes a fixed minor. There is some evidence to suggest that planar metrics are better behaved than general metrics with respect to $\ell_1$-embeddability. In an interesting recent development, Rao [33] has given an $O(\sqrt{\log n})$-distortion embedding of $n$-point planar metrics into $\ell_1$, while the lower bound for general metrics is $\Omega(\log n)$. This result, and the decomposition lemma of [22] on which it is based, attest to the special structure of planar metrics. Further evidence for this is provided by Konjevod et al., who have shown that any planar metric can be embedded with a distortion of $O(\log n)$ into a distribution over dominating tree metrics [24], while the best known upper bound for general metrics is still $O(\log n \log \log n)$ [5].

Despite this promise, current techniques are apparently inadequate to resolve the above conjecture. For embeddings into $\ell_1$, a celebrated result of Bourgain [9] tells us that any metric supported on an $n$-vertex graph (i.e., any metric on $n$ points) can be embedded into $\ell_1$ with distortion $O(\log n)$; unfortunately, the embedding technique is not sensitive to the topology and incurs a $\Omega(\log n)$-distortion even for the metric generated by the unit-weighted path $P_n$. Similarly, the method of Konjevod et al. of finding distributions over dominating trees is limited by a lower bound of $\Omega(\log n)$ for embedding the $n \times n$ grid [1, 24]. Lastly, Rao gives embeddings into $\ell_1$ by first embedding into $\ell_2$, an approach that is limited by a lower bound of $\Omega(\sqrt{\log n})$ for embedding even series-parallel graphs into $\ell_2$ [27].

In this paper, we systematically explore how the topology of a graph affects the distortion incurred by $\ell_1$-embeddings of metrics supported on it. Using the intimate connection between $\ell_1$-embeddability of metrics supported on a graph and multicommodity flow problems defined on it, one can show that graphs all of whose metrics are isometrically embeddable into $\ell_1$ (i.e., embeddable with distortion 1) are exactly the graphs which exclude $K_{2,3}$ as a minor, which essentially corresponds to the class of outerplanar graphs. This fact, which rests on a theorem of Okamura and Seymour [29], is our starting point. As a natural next step, we consider the family of graphs which have $K_4$ as an excluded minor. These are graphs with treewidth 2, and essentially correspond to the familiar class of series-parallel graphs. Our first main result is an explicit $\ell_1$-embedding of these graphs with small constant distortion. This is the first natural family known to have a constant distortion strictly bigger than 1. In addition, our construction implies a simple polynomial time algorithm for finding a sparsest cut within a constant factor of optimal in series-parallel graphs. In a similar vein, we also show that any family of graphs with bounded Euler characteristic can be embedded into $\ell_1$ with constant distortion. The technique we use for these results is to explicitly construct a set of cut metrics whose sum approximates the original graph metric very closely. Cut metrics arise naturally in the study of $\ell_1$-embeddability since any $\ell_1$-embeddable metric$^2$ can be represented as a sum of cut metrics with non-negative coefficients, and vice versa [15].

We then go on to study the approximation of a metric by a probability distribution over (dominating) tree metrics. Since tree metrics are $\ell_1$-embeddable (and so are their non-negative combinations), this gives us an alternative to the cut metrics approach. Furthermore, embeddings based on such metrics have proved particularly easy to work with, and possess additional properties that have been exploited in devising approximation algorithms and online algorithms for many problems (see, e.g., [4, 6, 3, 19, 36, 10, 12]). It is natural to ask if we can obtain the above embeddability results for outerplanar and series-parallel graphs using these more restricted metrics. The answers

$^2$We shall use the unqualified term “$\ell_1$-embeddable” to mean “isometrically embeddable into $\ell_1$.”
are mixed. On the one hand, we show that this is possible for outerplanar graphs, at a small cost: we give an explicit embedding for such graphs into a distribution over dominating tree metrics with distortion 8 (compared to distortion 1 obtained using cuts). On the other hand, we prove a complementatory negative result by exhibiting a family of series-parallel graphs for which any distribution over dominating tree metrics must necessarily incur a distortion of $\Omega(\log n)$.

Thus we see that the tree metrics approach breaks down at a surprisingly early stage (even for graphs of treewidth 2), which suggests that such embeddings by themselves offer little hope for exploring the finer structure of $\ell_1$-embeddings. However, our results also indicate that combining dominating tree metrics with cut metrics is a potentially powerful technique. Indeed, the graphs which give the lower bound for tree embeddings mentioned above can be shown to have extremely simple $\ell_1$-embeddings using cuts. Combining these cut metric embeddings with tree embeddings in a careful fashion leads us to an alternative constant distortion embedding for series-parallel graphs.

The organization of the paper closely follows the above outline. After a short section containing some definitions and notation, we briefly illuminate the connection between flows and $\ell_1$-embeddings in Section 3. The embeddings of series-parallel graphs and graphs with small Euler number are described in Section 4. Finally, in Section 5, we present our positive and negative results on embeddings into tree distributions, as well as the alternative embedding for series-parallel graphs.

2 Definitions and notation

**Metrics.** Let $X$ be a set. A function $d : X \times X \to \mathbb{R}^+$ is called a semi-metric if it is symmetric, i.e., $d(x, y) = d(y, x)$ for all $x, y \in X$, and $d(x, x) = 0$ for all $x \in X$, and also satisfies the triangle inequality, i.e., $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. If, in addition, $d(x, y) = 0$ holds only when $x = y$, then $d$ is a metric. In this paper, we shall only consider finite semi-metrics. The number of points will usually be denoted by $n$. Without risk of confusion, the distinction between metrics and semi-metrics may sometimes be blurred. For more details on many of the metric concepts used here, see the book of Deza and Laurent [15].

Given two metric spaces, $(V, \nu)$ and $(W, \mu)$, and a map $f : V \to W$, define the following quantities:

$$\|f\| = \max_{x, y \in V} \frac{\mu(f(x), f(y))}{\nu(x, y)},$$

$$\|f^{-1}\| = \max_{x, y \in V} \frac{\nu(x, y)}{\mu(f(x), f(y))}.$$  

We say that $f$ has contraction $\|f^{-1}\|$, expansion $\|f\|$ and distortion $D(f) = \|f\| \cdot \|f^{-1}\|$. We say that $(W, \mu)$ $r$-approximates $(V, \nu)$ (or that the distortion between $\mu$ and $\nu$ is at most $r$) if there exists a map $f : V \to W$ with $D(f) \leq r$. Often we shall consider two distance functions $\mu$ and $\nu$ over the same vertex set $V$. In such cases, we shall assume that $f$ is the identity map. Also, $\mu$ will be said to dominate $\nu$ if for all $x, y \in V$, $\mu(x, y) \geq \nu(x, y)$.

Let $G = (V, E)$ be an undirected graph. A metric $(V, \mu)$ is supported on (or generated by) $G$ if it is the shortest path metric of $G$ w.r.t. some non-negative weighting of the edges $E$. Unless specified
otherwise, we shall assume that the edge-weights \( w(\cdot) \) satisfy \( w(e) = \mu(e) \), where \( \mu \) is the shortest-path metric of \( G \) with weights \( w \). Observe that if it is not the case, the edge \( e \) can be removed without affecting the metric; such an \( e \) will be called redundant.

For a set \( S \subseteq V \), the cut metric \( \delta_S \) on \( V \) is defined by \( \delta_S(x, y) = 1 \) if \( |S \cap \{ x, y \}| = 1 \), and \( \delta_S(x, y) = 0 \) otherwise. An important observation is that the \( \ell_1 \)-embeddable metrics on \( V \) are precisely those metrics which can be written as a sum of cut metrics on \( V \) with non-negative coefficients [15]. One implication of this is that if two metrics \( \mu_1 \) and \( \mu_2 \) on the same underlying set are \( \ell_1 \)-embeddable, then so is their sum \( \mu_1 + \mu_2 \).

Finally, we use the following simple observation throughout the paper: if each block (i.e., biconnected component) \( G_i \) of a graph \( G \) can be embedded into \( \ell_1 \) with distortion \( D_i \), then \( G \) can be embedded into \( \ell_1 \) with distortion \( \max_i D_i \). This immediately implies, in particular, that any metric supported on a tree \( T \) can be embedded isometrically into \( \ell_1 \). (For a more direct proof of this latter fact, let \( (S_e, \delta_e) \) be the cut obtained by deleting an edge \( e \) in \( T \); it can be verified that \( \mu = \sum_{e \in T} d_T(e) \cdot \delta_e \), is isometric to the tree metric \( d_T \) [15, Prop.11.1.4],)

**Multicommodity flows:** A multicommodity flow network \( (V, E, P) \) is specified by an undirected graph \( G = (V, E) \), where \( E \) is the set of edges along which flow can be routed, and a set \( P \) of unordered pairs of vertices in \( V \) between which demands can be placed. In the unrestricted case, where \( P \) consists of all pairs of vertices, we shall omit explicit mention of \( P \) and refer to the network simply as \( G = (V, E) \). Assigning non-negative capacities \( C \) to the graph edges \( E \) and demands \( D \) to the pairs \( P \) gives us a particular instance \((V, C, D)\) of the multicommodity flow problem. For background, see the survey by Shmoys [35].

The optimal solution to this problem is the maximum value \( \lambda \) such that there is a multicommodity flow \( f \) respecting the edge capacities that satisfies a \( \lambda \)-fraction of each demand. We shall refer to \( \lambda \) as \( \text{MaxFlow}(V, C, D) \). Its value (as well as an actual flow \( f \) which realizes it) can be found in polynomial time by linear programming.

A closely related problem is the sparsest cut problem, which entails finding a partition \((A, \overline{A})\) of \( V \) that minimizes the ratio

\[
\kappa(A) = \frac{\text{Capacity}(A, \overline{A})}{\text{Demand}(A, \overline{A})} = \frac{C \cdot \delta_A}{D \cdot \delta_A}.
\]

(To make sense of the inner products, note that \( C, D \) and the cut metric \( \delta_A \) can all be viewed as elements of the vector space \( \mathbb{R}^n \).) We shall refer to \( \kappa = \min_A \kappa(A) \) as \( \text{MinCut}(V, C, D) \).

In the sequel it will be convenient to use the following identities (see, e.g., [26] or [15, page 135] for the proofs):

\[
\text{MaxFlow}(V, C, D) = \min_{\delta \in M(V)} \frac{C \cdot \delta}{D \cdot \delta} ; \quad \text{MinCut}(V, C, D) = \min_{\delta \in M_1(V)} \frac{C \cdot \delta}{D \cdot \delta}, \quad (2.1)
\]

where \( M(V) \) is the set (in fact, a convex cone) of all metrics over \( V \), and \( M_1(V) \) is the set (again, a convex cone) of all \( \ell_1 \)-embeddable metrics over \( V \). As \( M_1(V) \subset M(V) \) (the inclusion being strict for \( V \) of size \( \geq 5 \)), it is always the case that \( \text{MaxFlow} \leq \text{MinCut} \).
In contrast with the case when there is just one commodity, the MinCut is not equal to the MaxFlow in general. The ratio $\gamma \geq 1$ between the MinCut and the MaxFlow is called the gap of the instance $(V, C, D)$. From the computational point of view, computing the value of the MinCut (and hence also the value of $\gamma$) is an NP-hard problem.

**Graphs and Minors:** An outerplanar graph $G$ is a planar graph with an embedding in the plane so that every vertex lies on the outer (unbounded) face. A series-parallel graph $G = (V, E)$ with terminals $s, t \in V$ is either a single edge $\{s, t\}$, or a series combination or a parallel combination of two series-parallel graphs $G_1$ and $G_2$ with terminals $s_1, t_1$ and $s_2, t_2$. The series combination of $G_1$ and $G_2$ is formed by setting $s = s_1$, $t = t_2$ and identifying $s_2 = t_1$; the parallel combination is formed by identifying $s = s_1 = s_2$, $t = t_1 = t_2$.

The graph $G = (V, E)$ has an $H$-minor if there exists a sequence of edge-deletion and edge-contraction operations on $G$ which results in a graph $G'$ that is isomorphic to $H$. Note that each vertex of $G'$ corresponds to a (connected) set of vertices of $G$ which were contracted to it. For $U \subseteq V$, we say that $G$ has an $H$-minor w.r.t. $U$ if it has an $H$-minor $G'$ such that for every vertex of $G'$, the corresponding set of vertices of $G$ contains a vertex from $U$. Finally, we say that $G$ is $H$-free (w.r.t. $U$) if it has no $H$-minor (w.r.t. $U$).

It is well known that $K_4$-free graphs are those whose blocks are series-parallel graphs [16, p.185], and that $K_{2,3}$-free graphs are those whose blocks are either outerplanar or isomorphic to $K_4$ [16, p.81].

Finally, the Euler number of an undirected connected graph $G$ is defined as $\chi(G) = |E(G)| - |V(G)| + 1$. (Throughout this paper, the symbol $\chi(G)$ denotes the Euler number and *not* the chromatic number.)

### 3 Multicommodity flows, metrics and graphs

Multicommodity flows have long been an object of study in combinatorial optimization (see [18] for a historical survey). The classical theory was concerned mainly with the following question: Under what conditions on the flow network $(V, E, P)$ is the MaxFlow equal to the MinCut for every setting of capacities $C$ and demands $D$? As it turns out, this question is equivalent to the following question concerning the $\ell_1$-embeddability of metrics: What are the conditions on $(V, E, P)$ such that, for every metric $\mu$ supported on $G = (V, E)$, there exists an $\ell_1$-embeddable metric $\nu$ on $V$ such that $\mu$ dominates $\nu$, and $\mu = \nu$ on $P$? [34, Section 3]

In light of this equivalence, the classical results about flows (in cases where the gap $\gamma = 1$) have consequences for $\ell_1$-embeddability and *vice versa*. For instance, a well-known theorem due to Okamura and Seymour [29] says that if $G = (V, E)$ is a planar graph with outer face $F$, and $P$ consists only of pairs of vertices in $F$, then the MaxFlow and MinCut are equal for all instantiations of $C$ and $D$. Taking $G = (V, E)$ to be an outerplanar graph, letting $P$ consist of all pairs in $V$ and using the above equivalence, we can infer that all metrics supported on outerplanar graphs can be isometrically embedded into $\ell_1$. (See also [14] for a direct argument.)

To state this and other such results succinctly, let us introduce some notation. For a metric $\mu$, let
$c_1(\mu)$ be the minimum distortion between $\mu$ and $\rho$, where $\rho$ ranges over all $\ell_1$ metrics, and let $c_1(G)$ be the maximum value of $c_1(\mu)$ for all metrics $\mu$ supported on $G$. Hence, we have just seen that $c_1(G) = 1$ for every outerplanar graph $G$.

In fact, this turns out to be almost a characterization of graphs $G$ with $c_1(G) = 1$. The full picture is that $c_1(G) = 1$ iff $G$ is $K_{2,3}$-free. On the one hand, as mentioned earlier, each block of a $K_{2,3}$-free graph is either outerplanar or isomorphic to $K_4$, and a graph is $\ell_1$-embeddable iff each of its blocks is. We have already seen that outerplanar graphs are $\ell_1$-embeddable; it is also well known that the same holds for any metric on four points [15, Example 11.1.8]. Thus, for every $K_{2,3}$-free graph $G$, $c_1(G) = 1$. Conversely, it is well known that the metric of the unit-weighted $K_{2,3}$ is not $\ell_1$-embeddable [15, Example 6.3.2]. Now if $G$ has a $K_{2,3}$-minor, consider the sequence of edge contractions and deletions which turn $G$ into $K_{2,3}$. Assigning $\infty$ to each deleted edge, 0 to each contracted edge, and 1 to the remaining edges, we obtain a semi-metric supported on $G$ and coinciding (as a metric space) with that of the unit-weighted $K_{2,3}$. Thus, $c_1(G) \geq c_1(K_{2,3}) > 1$.

Hence we have the following characterization:

**Proposition 3.1** The class of graphs for which $c_1(G) = 1$ is exactly the class of $K_{2,3}$-free graphs.

Much recent research on multicommodity flows has been directed towards the case where equality does not hold, and to finding good bounds on the ratio $\gamma$ between the MinCut and the MaxFlow. This study was pioneered in the paper of Leighton and Rao [25], and the results presented there were extended in a long sequence of papers by several authors (see [35] for a detailed account). The best results known [26, 2] show that for any flow network $(V, E, P)$, the gap between the MaxFlow and the MinCut can never be more than $O(\log |P|)$, and hence $O(\log n)$. This bound is tight when $G = (V, E)$ is a constant-degree expander, all edge capacities are unity and there is unit demand between all pairs of vertices. Better results have been obtained for planar graphs, showing that in such graphs the gap $\gamma$ never exceeds $O(\sqrt{\log n})$ [33], and in fact is bounded by a constant in the special case of uniform demands [22].

An intimate relationship between the gap $\gamma$ and $c_1(G)$ holds even in the case where the MaxFlow is not equal to the MinCut, and provides a compelling motivation for studying the quantity $c_1(G)$.

**Theorem 3.2** For any graph $G = (V, E)$, the worst possible gap $\gamma$ attained by a multicommodity flow problem on $G$ is exactly $c_1(G)$.

**Proof:** The direction $\gamma \leq c_1(G)$ was shown already in [26]. Indeed, by definition of $c_1$, for every metric $\mu$ supported on $G$, there exists an $\ell_1$-embeddable metric $\delta$ which distorts $\mu$ by at most $c_1(G)$. But then, by definition of distortion, $\frac{C \delta}{D \delta} \leq c_1(G) \frac{C \mu}{D \mu}$, and in view of (2.1) we are done.

For the other, apparently new, direction $\gamma \geq c_1(G)$, it will be convenient to use an equivalent dual definition of $c_1(\mu)$ for a metric $\mu$ on $V$:

$$c_1(\mu) = \max \left( \frac{D \cdot \mu}{C \cdot \mu} \right),$$

where the maximum is taken over all non-negative vectors $C, D$ indexed by ordered pairs of vertices of $V$ which satisfy the restriction $\frac{D \delta}{C \delta} \leq 1$ for any $\ell_1$-embeddable metric $\delta$ on $V$. The proof of this equality follows from general facts about convex cones, and is deferred to the appendix.
By this dual definition, there exists a metric $\mu$ supported on $G$, and non-negative vectors $C, D \subseteq \mathbb{R}^{(|V|)}$, such that $D \mu = c_1(G)$, while for any $\ell_1$-embeddable metric $\delta$ we have $D \delta = c_1(G)$. First we claim that, without loss of generality, one may assume that $C$ vanishes outside $E(G)$. Indeed, assume that for some pair of vertices $(i, k) \notin E(G)$, the value $C(i, k)$ is strictly positive. Since $\mu$ is supported on $G$, there exist edges $c_1 = (j_0, j_1), c_2 = (j_1, j_2), \ldots, c_q = (j_{q-1}, j_q)$ in $G$ such that $j_0 = i, j_q = k$ and $\mu(j_0, j_q) = \mu(j_0, j_1) + \ldots + \mu(j_{q-1}, j_q)$. Define a new vector $C'$ by

$$C'(i, k) = 0,$$
$$C'(j_{r-1}, j_r) = C(j_{r-1}, j_r) + C(i, k) \quad \text{for each } r = 1, 2, \ldots q, \text{ and}$$
$$C'(u, v) = C(u, v) \quad \text{otherwise}.$$

Now, the pair $C', D$ can replace the pair $C, D$ in the above definition of $c_1(G)$. Clearly, for any metric $\delta$ on $V$ we have $C' \cdot \delta \geq C \cdot \delta$; in particular, for any $\ell_1$-embeddable $\delta$ we have $(D \cdot \delta)/(C' \cdot \delta) \leq (D \cdot \delta)/(C \cdot \delta) \leq 1$, as required by (3.2). On the other hand, for $\mu$, the “worst” metric supported on $G$, we have the equality $C' \cdot \mu = C \cdot \mu$, and thus $(D \cdot \mu)/(C' \cdot \mu) = (D \cdot \mu)/(C \cdot \mu) = c_1(G)$. Repeating this updating procedure for all non-edges of $G$, we arrive at a vector $C$ that vanishes outside $E(G)$.

Employing such a pair $C, D$ and bearing in mind the definitions of MinCut and MaxFlow given in (2.1), we conclude that

$$\gamma \geq \gamma(V, C, D) = \frac{\text{MinCut}(V, C, D)}{\text{MaxFlow}(V, C, D)} \geq \frac{\min_{\delta \in M_{\ell_1}(V)} (C \cdot \delta)/(D \cdot \delta)}{(C \cdot \mu)/(D \cdot \mu)} \geq \frac{D \cdot \mu}{C \cdot \mu} = c_1(G).$$

Recall that by Proposition 3.1, the graphs for which $c_1(G) = 1$ are exactly the $K_{2,3}$-free graphs. It is no coincidence that this characterization involves excluded minors. Observe that the graph-theoretic function $c_1$ is minor-monotone, i.e., if $H$ is a minor of $G$ then $c_1(G) \geq c_1(H)$. Indeed, edge deletion corresponds to assigning the edge the value $\infty$, while edge contraction corresponds to assigning it the value $0$. The principal consequence of this observation is that $\mathcal{F}_c$, the family of all graphs $G$ with $c_1(G) \leq c$, is minor-closed for any $c$. Hence, by a celebrated theorem of Robertson and Seymour, any $\mathcal{F}_c$ can be characterized in terms of forbidden minors (see, e.g., [16, Cor.12.5.3]).

Another consequence of monotonicity of $c_1(G)$ is that the set $\{c_1(G)\}$ contains no infinite descending sequence. Indeed, assume that $c_1(G_1) > c_1(G_2) > c_1(G_3) > \ldots$ is an infinite descending sequence. By a theorem of Robertson and Seymour, there must exist $G_i$ and $G_j$ with $j > i$ such that $G_i$ is a minor of $G_j$ (see, e.g., [16, Thm.12.5.2]), contradicting the monotonicity of $c_1$. In particular, every point of $\{c_1(G)\}$ contains a unique “next to the right” point. Currently, we only know that the smallest point of this set is 1, and the second smallest is $c_1(K_{2,3})$, which can be shown to be $4/3$.

An intriguing conjecture, and one of the main motivations behind this paper, is that for any non-trivial minor-closed family $\mathcal{F}$ of graphs, there exists a constant $c_{\mathcal{F}} \geq 1$ such that for all $G \in \mathcal{F}$, $c_1(G) \leq c_{\mathcal{F}}$.

The results in the next section provide some evidence in support of this conjecture. We consider the next natural minor-closed class of graphs containing $K_{2,3}$, namely the class of series-parallel
graphs, and show that they are $\ell_1$-embeddable with constant distortion. In addition, we bound the distortion $c_1(G)$ of a graph in terms of its Euler characteristic alone, and thus establish an infinite sequence of natural minor-closed families with constant distortion, namely those with bounded Euler characteristic.

4 Constant-distortion embeddings for some graph families

In this section, we shall present explicit constant-distortion embeddings into $\ell_1$ of the natural minor-closed families of series-parallel graphs, and of graphs with bounded Euler characteristic. These are the first non-trivial results exhibiting (necessarily) non-isometric embeddings of graph families with constant distortion.

4.1 Series-parallel graphs

Our goal will be to show that any metric supported on a series-parallel graph is embeddable in $\ell_1$ with constant distortion. In fact, our argument is presented for the slightly more general class of treewidth-2 graphs, i.e., graphs whose blocks are series-parallel graphs. Recall that this is a minor-closed family with $K_4$ as the excluded minor. We have not attempted to achieve the best possible constant distortion, which we believe is rather less than the value of (just under) 14 shown here.

**Theorem 4.1** Let $G = (V, E)$ be a weighted graph with treewidth 2, and let $\mu = \mu_G$ be the metric induced by the edge weights of $G$. Then there exists an $\ell_1$-embeddable metric $\tilde{\mu}$ and a constant $c < 14$ such that for every $u, v \in V$,

$$\frac{1}{c} \mu(u, v) \leq \tilde{\mu}(u, v) \leq \mu(u, v).$$

Moreover, this embedding preserves the length of edges, i.e., for every $(u, v) \in E$, $\tilde{\mu}(u, v) = \mu(u, v)$. Finally, $\tilde{\mu}$ can be computed in polynomial time.

Before proving the theorem, let us briefly discuss some properties of treewidth-2 graphs and the metrics generated by them. According to one of the many alternative definitions, treewidth-2 graphs can be constructed using the following composition procedure. Start with a single edge $e_0$, and repeatedly attach a single new vertex to the endpoints of an already existing edge (which we call the parent edge of the vertex); finally, after all the vertices have been attached, remove an arbitrary subset of the edges. We shall consider a weighted treewidth-2 graph $G$ together with the sequence of intermediate weighted graphs $G^2, G^3, \ldots, G^n = G$ occurring during its composition, where $G^2$ is the initial edge $e_0$. Each new edge $e = (u, v)$ will be endowed with weight $\mu(u, v)$, where $\mu$ is the metric induced by $G$. Observe that, w.l.o.g., we may assume that no edges are removed in the second stage of the construction, since removing a non-essential edge $e$ (one with weight $\mu(e)$) has no effect on $\mu$.

The manner in which $G$ was constructed implies that the metric $\mu^i$ induced by an intermediate graph $G^i$ on $V(G^i) \subseteq V(G)$ agrees with $\mu$ restricted to these vertices, i.e., $\mu^i = \mu|_{V(G^i)}$. A closer
look at the structure of $G$ reveals more information about $\mu$. Let us define the notions of ancestor and related edges of a vertex. The definition is recursive: the ancestor edges of $x \in V(G)$ include the parent edge $e = (s, t)$ of $x$, and the ancestor edges of $s$ and $t$. The first edge $e_0$ is an ancestor edge of both its endpoints, and thus of all $x$ in $V(G)$. A related edge of a vertex is an edge both of whose endpoints lie either on ancestor edges of $x$, or coincide with $x$. In particular, all ancestor edges of $x$ are also related edges of $x$.

An example is shown in Figure 4.1, in which the vertices were added in the order $x_1, x_2, x_3, x_4$. The parent edge of $x_4$ is $e_3$, its ancestor edges are $\{e_0, e_1, e_3\}$, while $\{(t, x_1), (x_1, x_3), (s, x_4), (x_3, x_4)\}$ are its related non-ancestor edges.

Let $e$ be an ancestor edge of $x$. Define $G^{x,e}$, a subgraph of $G$, as the union of all the related edges of $x$ which were introduced after $e$, plus edge $e$ itself. (For example, in Figure 4.1 the graph $G^{x_4,e_3}$ is the subgraph induced by the vertices $\{s, x_1, x_3, x_4\}$. The subgraph $G^{x,e}$ has a particularly simple structure: it is constructed by starting from $e$, marking it, and repeatedly attaching a single new vertex to the endpoints of the currently marked edge, upon which the marked edge is unmarked and one of the newly added edges is marked. The order of composition of $G^{x,e}$ is induced by that of $G$. The graph $G^{x,e}$ will simplify our later analysis; for the moment, observe that the distance between any pair of vertices in $G^{x,e}$ is equal to their original distance in $G$.

For a pair of vertices $x, y$, the last common ancestor edge $f = (s, t)$ of $x, y$ is the common ancestor edge of $x$ and $y$ which was added last in the composition of $G$. When neither $x$ nor $y$ lies on an ancestor edge of the other, two possibilities may occur: either $f$ separates $x$ and $y$ (i.e., every $x$-$y$ path passes through either $s$ or $t$), or there exists a vertex $q$ whose parent edge is $f$, such that $(s, q)$ is an ancestor edge of $x$ (but not of $y$) while $(t, q)$ is an ancestor edge of $y$ (but not of $x$).

We are now ready to embark on the proof of the theorem.

**Proof of Theorem 4.1:** We start with the inductive construction of the approximating metric $\tilde{\mu}$. The construction follows the composition procedure for $G$, first defining $\tilde{\mu}$ on $G^2$, then extending it to $G^3, G^4$, etc. in turn. In the base case, $G^2$ is a single edge $e_0 = (a, b)$, and we set $\tilde{\mu}(a, b) = \mu(a, b)$. For the inductive step, we assume that $\tilde{\mu}$ is already defined on $V(G^{i-1})$. Assume also that $G^i$ is obtained from $G^{i-1}$ by attaching a new vertex $x$ to the endpoints of the edge $(s, t)$. Let

$$\delta = \frac{\mu(x, s) + \mu(x, t) - \mu(s, t)}{2}; \quad P_s = \frac{\mu(x, t) - \mu(x, s) + \mu(s, t)}{2\mu(s, t)}; \quad P_t = \frac{\mu(x, s) - \mu(x, t) + \mu(s, t)}{2\mu(s, t)}.$$
Now, the value of $\tilde{\mu}(x, \cdot)$, where $\cdot$ stands for any vertex of $G^{i-1}$, is defined as

$$\tilde{\mu}(x, \cdot) = \delta + P_s \tilde{\mu}(s, \cdot) + P_t \tilde{\mu}(t, \cdot). \quad (4.3)$$

The definition of $\tilde{\mu}$ immediately implies that it is computable in polynomial time.

The argument that $\tilde{\mu}$ is $\ell_1$-embeddable is inductive. The base case is that $\tilde{\mu}$ on $G^2$ is trivially $\ell_1$-embeddable. For the inductive step, observe that $\tilde{\mu}$ on $G^i$ is a positive linear combination of three metrics: the cut metric $\delta_{[x]}$ (with coefficient $\delta$), the metric $\tilde{\mu}$ on $G^{i-1}$ with $x$ at distance 0 from $s$ (with coefficient $P_s$), and the metric $\tilde{\mu}$ on $G^{i-1}$ with $x$ at distance 0 from $t$ (with coefficient $P_t$). The cut metric is $\ell_1$-embeddable; $\tilde{\mu}$ on $G^{i-1}$ is $\ell_1$-embeddable by the induction hypothesis, and identifying the vertex $x$ with either $s$ or $t$ does not affect this. Thus, by induction, the restriction of $\tilde{\mu}$ to each $G^i$ (and hence to $G^n = G$) is a sum of $\ell_1$-embeddable metrics, and hence is $\ell_1$-embeddable.

The next fact to prove is that $\tilde{\mu}$ is dominated by $\mu$. Since $\mu$ is the shortest path metric of $G$, the expansion of $\tilde{\mu}$ is bounded by its expansion on the edges of $G$; thus it suffices to prove the stronger statement that every edge of $G$ maintains its length under $\tilde{\mu}$, i.e., for every $e = (u, v)$, $\tilde{\mu}(u, v) = \mu(u, v)$. This stronger statement is again established by an inductive argument. The claim obviously holds for $G^2$. Assume that the vertex $x$ is attached to the edge $(s, t) \in E(G^{i-1})$. By the inductive assumption, the claim holds for $G^{i-1}$, and in particular for $(s, t)$. Consider, e.g., the new edge $(x, s)$; by (4.3), $\tilde{\mu}(x, s) = \delta + P_t \tilde{\mu}(s, t) = \delta + P_t \mu(s, t)$ which, by definition of $\delta$ and $P_t$, equals $\mu(x, s)$.

Bounding the contraction of $\tilde{\mu}$ will be the hardest part of the proof. In preparation for this, let us give an equivalent but more intuitive “backwards” description of $\tilde{\mu}$. We envisage the process of constructing $\tilde{\mu}$ as starting from the final vertex, and collapsing the current “last” vertex onto one of the endpoints of its parent edge. More precisely, if the edge $(s, t)$ is the parent of $x$, we remove the cut metric corresponding to $x$ (with weight $\delta$), and then collapse the vertex $x$ onto either $s$ or $t$, with probabilities $P_s$ and $P_t$ respectively. (Note that $P_s$ and $P_t$ sum to 1, and both are non-negative by the triangle inequality.) Upon reaching $G^2$, we simply remove the corresponding cut metric, thus collapsing the entire graph to a single point. The metric $\tilde{\mu}$ is just the expected sum of the (weighted) cut metrics removed in this process. In what follows, we shall make repeated use of this view of $\tilde{\mu}$ as the expected result of a random process.

The bound we will prove on the contraction of $\tilde{\mu}$ is stated in the following lemma.

**Lemma 4.2** Let $x$ and $x^*$ be any two vertices of $G$. Then, for any $\xi \in (\frac{1}{2}, 1)$, we have

$$\tilde{\mu}(x, x^*) \geq \frac{(1 - \xi)(2\xi - 1)}{1 + \xi} \mu(x, x^*).$$

Theorem 4.1 follows at once from this lemma: we simply choose $\xi$ optimally to be $\sqrt{3} - 1$, and conclude that the contraction (and hence the distortion) of $\tilde{\mu}$ is at most 13.92.

We will split the proof of Lemma 4.2 into two cases:

**Case (i):** $x^*$ lies on an ancestor edge of $x$.

**Case (ii):** Neither $x$ nor $x^*$ lies on an ancestor edge of the other.
**Proof of Lemma 4.2, Case (i):** In this case \( x^* = s \) lies on an ancestor edge \( e = (s, t) \) of \( x \). Consider the graph \( G^{x,e} \) as defined above, and let \((s_1, t_1), \ldots, (s_k, t_k) = (s, t)\) be the sequence of ancestor edges of \( x \) up to the edge on which \( s \) lies. (See Figure 4.2.) For convenience, set also \( s_0 = t_0 = x \). For \( 1 \leq i \leq k \), define

\[
L_i = \mu(s_i, t_i); \quad \alpha_i = \mu(s_{i-1}, s_i); \quad \beta_i = \mu(t_{i-1}, t_i).
\]

Note that for each \( i \geq 2 \), either \( t_{i-1} = t_i \) with \( \beta_i = 0 \), or \( s_{i-1} = s_i \) with \( \alpha_i = 0 \).

Denote by \( P_i \) (resp., \( P_t \)) the probability (under the random-process definition of \( \tilde{\mu} \)) that, when \( x \) collapses to the edge \((s, t)\), it collapses onto \( s \) (resp., \( t \)). Let \( \Delta \) be the expected sum of the weights of the cuts removed under all collapses of \( x \) up to and including this time. Then we have

\[
\tilde{\mu}(x, s) = \Delta + P_t \mu(s, t),
\]

and therefore, by the edge preservation property of \( \tilde{\mu} \),

\[
\tilde{\mu}(x, s) = \Delta + P_t \mu(s, t) = \Delta + P_t \mu(s, t).
\]

(4.4)

Note also that not only is the actual distance \( \mu(x, s) \) equal in \( G \) and in \( G^{x,e} \), but the same holds for the approximated distance \( \tilde{\mu}(x, s) \); this is clear from (4.4) since the quantities \( \Delta \) and \( P_t \) must be equal in \( G \) and in \( G^{x,e} \). Thus in what follows we may restrict our attention to the subgraph \( G^{x,e} \).

Now let \( P^i_s \) (resp., \( P^i_t \)) be the probability that, when \( x \) collapses to the edge \((s_i, t_i)\), it collapses onto \( s_i \) (resp., \( t_i \)), and let \( \Delta^i \) be the expected sum of the weights of the cuts removed under all collapses of \( x \) up to and including this time. Assume also that \( t_i = t_{i-1} \) while \( s_i, s_{i-1} \) are distinct, as in Figure 4.2. (The other case is handled symmetrically.) The following claim establishes three inequalities relating the value of \( \tilde{\mu}(x, s_i) \) to the values of \( \tilde{\mu}(x, s_{i-1}) \) and \( \tilde{\mu}(x, t_{i-1}) \).

**Claim 4.3** Let \( \xi \in (\frac{1}{2}, 1) \). Then, in the above situation,

(a) If \( P^i_s \geq \xi \), then \( \tilde{\mu}(x, s_i) \geq \tilde{\mu}(x, s_{i-1}) + (2\xi - 1)\alpha_i \).

(b) If \( P^i_t \geq \xi \), then \( \tilde{\mu}(x, s_i) \geq \tilde{\mu}(x, t_{i-1}) + (2\xi - 1)\beta_i \).

(c) Otherwise, if \( 1 - \xi \leq P^i_s \leq \xi \), then \( \tilde{\mu}(x, s_i) + \frac{2\xi}{1-\xi}(\Delta^i - \Delta^{i-1}) \geq \tilde{\mu}(x, s_{i-1}) + \alpha_i \).
Proof: The proof is elementary but somewhat technical. Arguing as in the derivation of (4.4), we obtain
\[
\tilde{\mu}(x, s_{i-1}) = \Delta^{i-1} + P^{i-1}_i L_{i-1}; \\
\tilde{\mu}(x, t_{i-1}) = \Delta^{i-1} + P^{i-1}_s L_{i-1}.
\] (4.5)

Keeping in mind the edge preservation property of \(\tilde{\mu}\), and conditioning on whether \(x\) collapsed onto \(s_{i-1}\) or \(t_{i-1}\), we can express \(\tilde{\mu}(x, s_i)\) as
\[
\tilde{\mu}(x, s_i) = \Delta^{i-1} + P^{i-1}_i L_i + P^{i-1}_s \alpha_i.
\] (4.6)

Performing a formal manipulation, we get
\[
\tilde{\mu}(x, s_i) = \Delta^{i-1} + P^{i-1}_i (L_i + \alpha_i) + (P^{i-1}_s - P^{i-1}_s) \alpha_i \\
\geq \Delta^{i-1} + P^{i-1}_i L_{i-1} + (P^{i-1}_s - P^{i-1}_s) \alpha_i = \tilde{\mu}(x, s_{i-1}) + (P^{i-1}_s - P^{i-1}_s) \alpha_i,
\]
where we have used the triangle inequality \(L_{i-1} \leq L_i + \alpha_i\), and (4.5). This implies (a).

Similarly,
\[
\tilde{\mu}(x, s_i) = \Delta^{i-1} + P^{i-1}_s (L_i + \alpha_i) + (P^{i-1}_i - P^{i-1}_i) L_i \\
\geq \Delta^{i-1} + P^{i-1}_s L_{i-1} + (P^{i-1}_i - P^{i-1}_i) L_i = \tilde{\mu}(x, t_{i-1}) + (P^{i-1}_i - P^{i-1}_i) L_i,
\]
implying (b).

In order to show (c), consider the change in \(\Delta\). Let \(\delta^{i-1}\) be the weight of the cut removed while collapsing \(s_{i-1}\) to \((s_i, t_i)\). Then
\[
\Delta^i - \Delta^{i-1} = P^{i-1}_s \delta^{i-1} = P^{i-1}_s \cdot \frac{\alpha_i + L_i - L_{i-1}}{2}.
\]
Substituting this expression for the value of \((\Delta^i - \Delta^{i-1})\), and using (4.6) and (4.5), we get
\[
\tilde{\mu}(x, s_i) \geq \tilde{\mu}(x, s_{i-1}) + \alpha_i.
\]

We are now in a position to bound \(\tilde{\mu}(x, s)\) from below in terms of \(\mu(x, s)\). For this purpose, we will construct a path between \(x\) and \(s\) in \(G^{x,e}\), and show that every edge on this path makes a substantial contribution to \(\tilde{\mu}(x, s)\). Since the length of the path is at least \(\mu(x, s)\), this will yield the desired lower bound.

The path \(\Pi\) from \(s = s_k\) to \(x\) in \(G^{x,e}\) will be defined as follows. Assume we have already constructed some initial segment of \(\Pi\), and have reached an endpoint of the edge \((s_i, t_i)\), but have not yet reached the edge \((s_{i-1}, t_{i-1})\). Assume also, w.l.o.g., that \(s_i, t_i\) are again situated as in Figure 4.2; the other case is treated in a symmetrical manner. Then we must have reached \(s_i\). Consider the value of \(P^{i-1}_i\) defined above. If \(P^{i-1}_i > \xi\), we add to \(\Pi\) the edge \((s_i, t_{i-1})\) of length \(L_i\) and continue; otherwise,
we add to $\Pi$ the edge $(s_i, s_{i-1})$ of length $a_i$ and continue. Upon reaching $(s_1, t_1)$, we add the edge connecting $x$ to $(s_1, t_1)$ to complete the path $\Pi$.

Clearly, $\Pi$ is a well-defined path from $s = s_k$ to $x$ in $G^{x,e}$. Moreover, by our choice of $\Pi$ and the preceding analysis (i.e., Claim 4.3), if $\Pi$ is $\langle s_k = \pi_0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \cdots \rightarrow \pi_m = x \rangle$, then for every edge $(\pi_{j-1}, \pi_j) \in \Pi$ we have

$$\tilde{\mu}(x, \pi_j) - \tilde{\mu}(x, \pi_{j-1}) + \frac{2\xi}{1 - \xi} (\Delta^{\pi_j} - \Delta^{\pi_{j-1}}) \geq (2\xi - 1) \cdot \mu(\pi_{j-1}, \pi_j),$$

where, with a slight abuse of notation, $\Delta^{\pi_j}$ stands for $\Delta^r$ where $r$ is the smallest index such that $\pi_j \in (s_r, t_r)$. (Observe that $(\Delta^{\pi_j} - \Delta^{\pi_{j-1}}) \geq 0$, so we may safely add this term for all $j$.)

Summing up these expressions, we arrive at

$$\tilde{\mu}(x, s_k) + \frac{2\xi}{1 - \xi} \Delta^k \geq (2\xi - 1) \cdot (\text{the } \mu\text{-length of } P) \geq (2\xi - 1) \mu(x, s_k). \quad (4.7)$$

Since clearly $\tilde{\mu}(x, s_k) \geq \Delta^k$, this completes the proof of Case (i) of Lemma 4.2. ■

**Proof of Lemma 4.2, Case (ii):** In this case, neither $x$ nor $x^*$ lies on an ancestor edge of the other. Let $(s, t)$ be the last common ancestor edge of $x$ and $x^*$. As mentioned before, there are two possibilities. The first is that $(s, t)$ separates $x$ and $x^*$. The second is that there is a triangle $T = (s, q, t)$ such that $(s, q)$ is an ancestor edge of $x$ but not of $x^*$, $(t, q)$ is an ancestor edge of $x^*$ but not of $x$, and both $(s, q)$ and $(t, q)$ separate $x$ from $x^*$.

We start with the analysis of the first possibility. Let $P_s$ (resp., $P_t$) denote the probability that when $x$ collapses to $(s, t)$, it collapses onto $s$ (resp., $t$); the probabilities $P_s^*$ (resp., $P_t^*$) are the corresponding values for $x^*$. Also, let $\Delta$ (resp., $\Delta^*$) be the expected value of the sum of the weights of cut metrics removed in the process of collapsing $x$ (resp., $x^*$) to the edge $(s, t)$. By the random process definition of $\tilde{\mu}$, the collapses of $x$ and of $x^*$ proceed independently of each other; keeping in mind that $\tilde{\mu}$ is preserved on edges, we get

$$\tilde{\mu}(x, x^*) = \Delta + \Delta^* + (P_s P_t^* + P_t P_s^*) \mu(s, t). \quad (4.8)$$

Moreover, it can be easily verified that

$$P_s P_t^* + P_t P_s^* \geq \frac{1}{2} \min \{ P_s + P_s^*; P_t + P_t^* \}. \quad (4.9)$$

Substituting this into (4.8), assuming w.l.o.g. that the minimum is attained at $t$, and using (4.4), we get

$$\tilde{\mu}(x, x^*) \geq \frac{1}{2} (\tilde{\mu}(x, s) + \Delta) + \frac{1}{2} (\tilde{\mu}(x^*, s) + \Delta^*). \quad (4.10)$$

However, adding the inequality (4.7) times the positive constant $\zeta = \frac{1 - \xi}{1 + \xi}$ to the inequality $\mu(x, s) - \Delta \geq 0$ times the positive constant $(\frac{1}{2} - \zeta)$, gives

$$\frac{1}{2} (\tilde{\mu}(x, s) + \Delta) \geq \frac{(1 - \xi)(2\xi - 1)}{1 + \xi} \mu(x, s).$$
An analogous bound holds for \( \tilde{\mu}(x^*, s) \). These two bounds, together with (4.10) and the triangle inequality \( \mu(x, x^*) \leq \mu(x, s) + \mu(s, x^*) \), imply the Lemma when the first possibility occurs.

We now look at the second possibility, i.e., when there is the triangle \( T = (s, t, q) \). To compute the values of \( \mu(x^*, x) \) and \( \tilde{\mu}(x^*, x) \) in the original graph \( G \), it suffices to look instead at the random process restricted to the graph \( H \) obtained by taking the graphs \( G^x(s, q) \) and \( G^x(t, q) \) and attaching them to the triangle \( T = (s, q, t) \). (This follows by the same reasoning as in Case (i), when we argued that the values of \( \mu(x, s) \) and \( \tilde{\mu}(x, s) \) in \( G \) could be computed by restricting our attention to \( G^{x^*} \).)

The random process goes as follows: the graph \( H \) is first collapsed onto \( T \), the vertex \( q \) is then collapsed onto either \( s \) or \( t \), and finally the resulting \( \{s, t\} \)-cut is removed. Let us define a new random process, which collapses \( H \) onto \( T \) as before, but then collapses \( t \) onto \( (s, q) \) and removes the resulting \( \{s, q\} \)-cut. Our claim is that the value of \( \tilde{\mu}(x, x^*) \) is the same in both processes. Indeed, the two processes differ only in the final step, and it is simple to check that, given a triangle, the random process generates the same metric regardless of which vertex is collapsed onto its opposite edge.

Now, in this new order that we have introduced, the last common ancestor edge of \( x, x^* \) is \( (s, q) \), and this edge separates \( x \) and \( x^* \). At this point, the argument for the first possibility applies, and the claim follows.

This completes the verification of both cases in the proof of Lemma 4.2, and hence the proof of Theorem 4.1.

Having proved the main theorem of this section, let us state some corollaries and observations.

Much of the complication in the proof arises from the need to account for both the cuts removed and the collapses made at each step. Let us consider for the moment the important special situation in which no cuts are removed, i.e., when the input series-parallel graph \( G \) has the property that for all \( x \), for all ancestor edges \( (s, t) \) of \( x \) we have \( \mu(x, s) + \mu(x, t) = \mu(s, t) \). (Observe that this property can be restated in a simpler form: for all \( x \), we have \( \mu(x, a) + \mu(x, b) = \mu(a, b) \), where \( a, b \) are the terminals of \( G \). We shall point out an interesting application of these graphs in Section 5.4.)

For such graphs a stronger version of Lemma 4.2 is true: namely, \( \tilde{\mu}(x, x^*) \geq \frac{1}{2} \mu(x, x^*) \). Moreover, the proof is much simpler than in the general setting. To see this, consider first Case (i) (when \( x^* = s \) lies on an ancestor edge of \( x \)); in this case we actually have that \( \tilde{\mu}(x, s) = \mu(x, s) \), and this follows directly from the definition of \( \tilde{\mu} \) using induction on the composition of \( G \). Indeed, assume that \( x \) is attached to \( (s_1, t_1) \), and the claim has already been established for \( s_1, t_1 \). By definition of \( \tilde{\mu} \),

\[
\tilde{\mu}(x, s) = \frac{\mu(x, s_1)}{\tilde{\mu}(s_1, t_1)} \cdot \tilde{\mu}(t_1, s) + \frac{\mu(x, t_1)}{\tilde{\mu}(s_1, t_1)} \cdot \tilde{\mu}(s_1, s).
\]

By the inductive hypothesis,

\[
\tilde{\mu}(t_1, s) = \mu(t_1, s) = \mu(t_1, s_1) + \mu(s_1, s) \quad \text{and} \quad \tilde{\mu}(s_1, s) = \mu(s_1, s).
\]

Combining the equations, we get \( \tilde{\mu}(x, s) = \mu(x, s) \) as claimed. Case (ii) of Lemma 4.2 can now be strengthened to \( \tilde{\mu}(x, x^*) \geq \frac{1}{2} \mu(x, x^*) \). This follows from (4.10), keeping in mind that \( \Delta = \Delta^e = 0 \) and using the stronger version of Case (i) given above. Thus, we can conclude:
Lemma 4.4 For the special series-parallel graphs described above, \( \frac{1}{2} \mu \leq \tilde{\mu} \leq \mu \).

Returning now to the gap \( \gamma \) in multicommodity flow instances, Theorems 3.2 and 4.1 imply:

Corollary 4.5 Let \( G = (V, E) \) be a graph with no \( K_4 \)-minor. Then, for every assignment of edge capacities \( C \) and demands \( D \) in \( G \), the gap \( \gamma = \text{MinCut}/\text{MaxFlow} \) is less than \( 14 \).

With the aid of a little graph-theoretic machinery, this corollary can be generalized as follows. The proof is somewhat orthogonal to our main development, and can be found in a separate paper [28].

Theorem 4.6 Let \( G = (V, E) \) be a graph, and let the set of demand pairs be a subset of pairs from \( U \), for some \( U \subseteq V \). If \( G \) contains no \( K_4 \)-minor w.r.t. \( U \), then for every assignment of edge capacities \( C \) and demands \( D \) in \( G \), the gap \( \gamma = \text{MinCut}/\text{MaxFlow} \) is less than \( 28 \).

4.1.1 Approximating the sparsest cut in series-parallel graphs

The iterative procedure used in the above proof can be exploited to find a near-optimal sparsest cut in series-parallel graphs in polynomial time. Previously, this result was known only for the special case of uniform demands [32, 30, 22]. Observe that Corollary 4.5 alone does not immediately imply the existence of a polynomial time procedure for finding a good cut.

Theorem 4.7 There is a polynomial time \( 14 \)-approximation algorithm for the Sparsest Cut problem on series-parallel graphs.

Proof Sketch: To approximate the MinCut in a series-parallel graph, we first solve the corresponding multicommodity flow problem, and find the metric \( \mu \) minimizing \( \frac{C \mu}{D \mu} \) (see the discussion following Theorem 3.2). By Theorem 4.1, we can find in polynomial time an \( \ell_1 \)-metric \( \tilde{\mu} \) that 14-approximates \( \mu \). Recall the manner in which \( \tilde{\mu} \) is built (see equation (4.3) and the description following it): at each step, it is a positive linear combination of three \( \ell_1 \)-metrics \( \tilde{\mu}_1, \tilde{\mu}_2 \) and \( \tilde{\mu}_3 \). Consequently, at least one of these metrics must yield a value \( \frac{C \tilde{\mu}}{D \tilde{\mu}} \), which is at most \( \frac{C \tilde{\mu}}{D \tilde{\mu}} \). Choosing this minimizing metric and continuing with the corresponding subgraph, we will eventually reach a point where the remaining metric is a cut metric. This cut achieves the desired approximation ratio.

4.2 Embedding graphs with few edges

Recall that for a graph \( G = (V, E) \), the Euler characteristic \( \chi(G) \) is defined as \( |E| - |V| + 1 \). It is easy to see that, for each \( c \in \mathbb{Z}^+ \), the family of graphs \( \mathcal{F}_c = \{ G \mid \chi(G) \leq c \} \) is minor-closed. The following theorem shows that graphs with low \( \chi(G) \) can be embedded with low distortion into \( \ell_1 \):

Theorem 4.8 A metric supported on an arbitrary graph \( G \) can be embedded into \( \ell_1 \) with distortion \( O(\log \chi(G)) \), where \( \chi(G) \) is the Euler characteristic of \( G \).
Proof: The embedding will be similar in flavor to that of Theorem 4.1, though much simpler. As before, we assume that $G$ is 2-connected; if not, we can apply the argument to each of its blocks. We also assume that $G$ is not a cycle, since the cycle metric embeds isometrically into $\ell_1$, as can be deduced from Proposition 3.1 (or for a direct proof see [26, Prop.5.10]).

Define an isolated path to be a maximal path in $G$, each of whose internal vertices has degree 2. Hence each of its endpoints has degree at least 3. Call an isolated path $B$ tight if its length is equal to the distance between its endpoints. We first decompose $d_G$, the shortest-path metric of $G$, into two simpler metrics: $\tilde{\mu}$, which is the shortest-path metric of a graph $G'$ with the same vertices and edges as $G$ but which has only tight isolated paths, and $\tilde{\mu}'$, which is a sum of cut metrics.

For this, let us consider a weighted cycle $C$, assuming that the weight of any edge is just its shortest-path length. Let $e = (u, v)$ be an edge on $C$. Since $C$ is $\ell_1$-embeddable, the metric $d_C$ can be written as a positive linear combination of cut metrics. Let $d_0$ be the sum of all those cuts that separate $u$ and $v$, and $d_1$ be the sum over the remaining cuts; clearly, $d_C = d_0 + d_1$. Observe that the sum of $d_0$-lengths of all the edges in $E(C) - \{e\}$ is necessarily exactly equal to the length of $e$, or, in other words, the length of the path $P = C - \{e\}$ under $d_0$ is equal to the length of $e$; note also that $d_0(e) = d_C(e)$. Concerning $d_1$, observe that no cut metric $\delta_S$ in $d_1$ separates $u$ and $v$, so we may assume w.l.o.g. that the corresponding set $S$ satisfies $S \subseteq V(C) - \{u, v\}$.

All this leads to a decomposition of $G$ into $G'$ plus an $\ell_1$ metric. Suppose $G$ has isolated paths that are not tight. To the endpoints $u$ and $v$ of each isolated path $B$, add an edge $e = (u, v)$ of length $d(u, v)$; this forms a cycle with $B$. The shortest path metric of each such cycle can be decomposed into $d_0$ and $d_1$ as above. Each of the cut metrics in $d_1$ naturally extends to the whole of $G$, and hence $d_1$, being their weighted sum, also extends to an $\ell_1$-embeddable metric on $G$. Call this $\tilde{\mu}'$. By the preceding discussion, $d_G = d_{G'} + \tilde{\mu}'$, where $G'$ has the same vertices and edges as $G$, but all isolated paths in $G'$ are now tight (as in $d_0$). This is the desired decomposition.

Since this phase involved no distortion, it suffices for the proof of the theorem to show that any graph $G$ with tight isolated paths can be embedded into $\ell_1$ with distortion $O(\log \chi(G))$. We will denote the length of an isolated path $B$ by $d(B)$.

Let $\tilde{G}$ be a minor (multigraph) of $G$ obtained by the following random procedure: for each isolated path $B$ with endpoints $u_B$ and $v_B$, choose a value $\tau_B$ uniformly and independently from the interval $[0, d(B)]$, and collapse all vertices in $B$ at distance less than $\tau_B$ from $v_B$ to this endpoint, and all the other vertices in $B$ to $u_B$. The length of the newly created edge $(u_B, v_B) \in E(\tilde{G})$ is defined as $d(B) = d_G(u_B, v_B)$, so that the distance between $u_B$ and $v_B$ remains unchanged. Clearly, the minimum degree of $\tilde{G}$ is now at least 3. Define $\tilde{\mu}(\cdot, \cdot) = \mathbb{E} [d_{\tilde{G}}(\cdot, \cdot)]$; being a convex combination of metrics, $\tilde{\mu}$ is a metric as well. We claim that $\tilde{\mu}$ closely approximates $d$.

Claim 4.9 For any two vertices $x, y$ of $G$, the expected distance $\tilde{\mu}$ between $x$ and $y$ in $\tilde{G}$ satisfies
\[
\frac{1}{4} d(x, y) \leq \tilde{\mu}(x, y) \leq d(x, y).
\]

Proof: Let us start with two simple observations. Firstly, if neither $x$ nor $y$ is an internal vertex of an isolated path, the distance between them remains the same, i.e., $\tilde{\mu}(x, y) = d(x, y)$. Furthermore, a simple calculation (involving the probability that $x$ and $y$ are collapsed to different endpoints of
$B$) shows that the same is true for any $x$ and $y$ belonging to the same isolated path $B$. Thus $\tilde{\mu}$ preserves the lengths of all the edges of $G$, and since $d$ is the shortest-path distance in $G$, we infer that $\tilde{\mu}$ is dominated by $d$.

Consider now the case when the vertices $x, y$ lie on different isolated paths $B$ and $B'$. Let $s, t$ be the endpoints of $B$, and $q, r$ the endpoints of $B'$. Define $P_q$ and $P_r$ to be the probabilities that $x$ is contracted to $s$ and $t$ respectively. $P_q$ and $P_r$ are defined similarly, with respect to $y$. Clearly,

$$P_s = \frac{d(x,t)}{d(s,t)}; \quad \text{and} \quad P_t = \frac{d(x,s)}{d(s,t)}.$$

The expressions for $P_q$ and $P_r$ are analogous. By the definition of $\tilde{\mu}$,

$$\tilde{\mu}(x,y) = P_q P_q \cdot d(s,q) + P_r P_r \cdot d(s,r) + P_q P_r \cdot d(t,q) + P_r P_r \cdot d(t,r)$$

$$= P_s \cdot [P_q d(s,q) + P_r d(s,r)] + P_t \cdot [P_q d(t,q) + P_r d(t,r)].$$

(4.11)

A scaled version of (4.9) together with the triangle inequality implies that

$$P_q d(s,q) + P_r d(s,r) \geq \frac{1}{2} \min \{ P_q [d(s,q) + d(s,r)] + d(s,r) : P_r [d(s,q)+d(s,r)] + d(s,q) \}$$

$$\geq \frac{1}{2} \min \{ P_q d(q,r) + d(s,r) : P_r d(q,r) + d(s,q) \}$$

$$= \frac{1}{2} \min \{ d(y,r) + d(s,r) : d(y,q) + d(s,q) \}$$

$$= \frac{1}{2} d(s,y).$$

Similarly, $P_q d(t,q) + P_r d(t,r) \geq \frac{1}{2} d(t,y)$. Substituting these inequalities into (4.11), and using the scaled version of (4.9) again, we conclude that

$$d(x,y) \geq \frac{1}{2} \{ P_q d(s,y) + P_r d(t,y) \} \geq \frac{1}{2} \cdot \frac{1}{2} d(x,y).$$

This completes the proof of the claim. \hfill \blacksquare

Thus $d$ is 4-approximated by $\tilde{\mu}$. To conclude the proof of the theorem, we show that $\tilde{\mu}$ can be embedded into $\ell_1$ with small distortion. Note that $\tilde{\mu}$ is a convex combination of semimetrics, all of which are supported on $G'$, the graph obtained from $G$ by replacing each isolated path by an edge. The distortion of embedding $\tilde{\mu}$ into $\ell_1$ is no more than that of $d_{G'}$, so it suffices to bound the latter.

But $G'$ has very few vertices. On the one hand, it has minimum degree $\geq 3$; on the other hand, it is a minor of $G$, and since taking minors cannot increase the Euler number, $\chi(G) \geq \chi(G')$. Let $n' = |V(G')|$, and $m' = |E(G')|$. By a degree argument, $m' \geq \frac{3}{2} n'$, implying $\chi(G) \geq \chi(G') \geq \frac{3}{2} n' + 1$. Consequently, $G'$ has at most $2\chi(G) - 2$ vertices, and hence $d_{G'}$ can be embedded into $\ell_1$ (e.g., using Bourgain’s technique [9]) with distortion $O(\log \chi(G))$. \hfill \blacksquare

5 Embodiments via tree metrics

The algorithms for $\ell_1$-embodiments described in the previous section were based on constructing an approximating set of cut metrics. A different approach for embedding a metric $(V, \mu)$ into $\ell_1$ is
to specify a probability distribution over trees containing $V$, such that the expected tree distance between any two vertices $x$ and $y$ in $V$ approximates $\mu(x, y)$ well. Since trees can be embedded isometrically into $\ell_1$, this also gives an $\ell_1$-embedding. Of particular interest are embeddings into distributions over dominating trees, in which the distance function in each tree dominates $\mu$. Finding low-distortion embeddings of this kind has consequences for the design of many approximation algorithms (e.g., [4, 3, 19, 36, 10, 12]) and online algorithms (e.g., [4, 6]). Formally:

**Definition 5.1** A metric $d_G$ supported on a graph $G$ is $\alpha$-probabilistically approximated by a distribution $\mathcal{D}$ over (dominating) trees if

1. each tree $T$ in the distribution $\mathcal{D}$ has $V(G) \subseteq V(T)$;
2. for all $x, y \in V$ and $T$ in the distribution, $d_T$ dominates $d_G$, i.e., $d_G(x, y) \leq d_T(x, y)$;
3. for all $x, y \in V$, the expected distance $\mathbb{E}_T[d_T(x, y)] \leq \alpha \cdot d_G(x, y)$.

In this paper we will use only spanning subtrees of $G$, and thus (1) and (2) will automatically be satisfied. Since the expansion is always maximal on the edges of $G$, condition (3) can be replaced by the more convenient

(3') for all edges $e = (x, y) \in E(G)$, the expected distance $\mathbb{E}_T[d_T(x, y)] \leq \alpha \cdot d_G(x, y)$.

We shall also refer to this approximation as an embedding of $d_G$ with distortion $\alpha$ into a tree distribution $\mathcal{D}$.

Distributions over trees were first studied by Karp, who showed that distances in the unweighted cycle $C_n$ can be $2(1 - \frac{1}{n})$-probabilistically approximated by a distribution over its subtrees [21]. The distribution is very simple: each possible spanning tree of $G$ is output with probability $1/n$. This is in sharp contrast to the deterministic case, where it can be shown that any tree (not necessarily a subtree) approximating the cycle has $\Omega(n)$ distortion [31]. This line of enquiry was further developed in several papers [1, 4, 5, 24, 11], where distributions over arbitrary dominating trees were considered. The state-of-the-art results show that any graph with $n$ vertices can be embedded into tree distributions with distortion $O(\log n \log \log n)$ [5]. In the special case where the graph excludes a $K_{s,s}$-minor, a distortion of $O(s^3 \log n)$ can be achieved [24]. In line with our general approach, we now study the embeddability of outerplanar and series-parallel graphs into tree distributions.

### 5.1 Tree embeddings for outerplanar graphs

The first result of this section shows that any metric supported on a $K_{2,3}$-free graph can be embedded into a tree distribution with distortion at most 8. Of course, we already know by Proposition 3.1 that such metrics are isometrically embeddable into $\ell_1$. However, that result says nothing about the stronger requirement that the embedding be a distribution over dominating trees. Both the main result of this section and the method used play an essential part in later, more difficult constructions (see, e.g., Section 5.4, and the recent [13]).

As usual, it suffices to embed only the biconnected components of the $K_{2,3}$-free graph, which are either $K_4$ or outerplanar. It is easy to verify that approximating any metric on $n$ points by its minimum-weight spanning tree incurs a distortion of at most $(n - 1)$, so any 4-point metric can be
embedded into a tree with distortion 3. Thus, it suffices to bound the distortion for 2-connected outerplanar graphs. As always, we assume w.l.o.g. that the length of any edge is equal to the distance between its endpoints.

We start with a composition procedure for outerplanar graphs which will form the basis for the embedding. Given such a graph $G$, one can define a sequence of outerplanar graphs $G_0, G_1, \ldots, G_t = G$, where $G_0$ is a path or a cycle, and the graph $G_i$ is obtained by attaching a path $P_i$ either to a single vertex $u_i$ on the outer face of $G_{i-1}$, or to the endpoints of an edge $e_i = \{u_i, v_i\}$ lying on the outer face of $G_{i-1}$. In the latter case, since the length of any edge is equal to the distance between its endpoints in $G$, the path $P_i$ is at least as long as $e_i$. This implies that the shortest-path metric of the graph $G_i$ coincides with the metric induced by $d_G$ on $V(G_i)$. Clearly, the composition of $G$ is completely specified by $G_0$ and the sequence of paths $\{P_i\}$.

Given an outerplanar graph $G$ with a specified composition procedure, the path $P_i$ is called slack if either $P_i$ is attached to a single vertex, or $P_i$ is attached to an edge $e_i$ and the length of $P_i$ is at least twice the length of $e_i$. A composition is called slack if all the paths $P_i$ in it are slack. We shall first provide an embedding procedure for an outerplanar graph $G$ assuming that $G$ has a slack composition, and then show how to extend this to all outerplanar graphs.

**Lemma 5.2** Given an outerplanar graph $G$ and a slack composition for it, $G$ can be embedded into a tree distribution $D$ with distortion at most 4.

**Proof:** The embedding is inductive and follows the composition. At stage $i$, we shall construct a random spanning tree $T_i$ of $G_i$ from a random spanning tree $T_{i-1}$ of $G_{i-1}$, while maintaining property (3') for $T_i$ with $\alpha = 4$; i.e., with $E[d_{T_i}(x, y)] \leq 4d_{G_i}(x, y)$ for all edges $(x, y) \in G_i$.

In the base case, if $G_0$ is a path, we do nothing. If it is a cycle, we randomly pick an edge $e$ of $G_0$ with probability proportional to its length, and delete it to get a random subtree of $G_0$. Let the length of $e$ be $l$, and the length of $G_0$ be $L$. The expected distance between the endpoints of $e$ in $T_0$ is

$$\left(\frac{l}{L}\right) \cdot (L - l) + \left(\frac{L - l}{L}\right) \cdot l \leq 2l,$$

satisfying property (3').

At stage $i$, we look at $P_i$. If it is attached to a single vertex $u_i$, we attach it to $T_{i-1}$ at $u_i$ to get $T_i$. Clearly, property (3') continues to hold for $T_i$. On the other hand, if $P_i$ is attached to an edge $e_i$, we randomly pick an edge $e$ from $P_i$ (again with probability proportional to the length of $e$) and set $T_i = T_{i-1} \cup (P_i - \{e\})$. It is clear that $T_i$ is a spanning tree of $G_i$. Let us show that property (3') is maintained. By the induction hypothesis, this is true for edges $(x, y)$ of $G_{i-1}$, since

$$E[d_{T_i}(x, y)] = E[d_{T_{i-1}}(x, y)] \leq 4d_{G_{i-1}}(x, y) = 4d_{T_i}(x, y).$$

Consider an edge $e = (x, y) \in P_i$; denote its length by $l$, and the length of $P_i$ by $L_i$. Furthermore, assuming that $P_i$ is attached at the edge $(u_i, v_i)$, denote $d_{G_{i-1}}(u_i, v_i)$ by $d$. The expected distance between $x$ and $y$ in $T_i$ is at most

$$\left(\frac{l}{L_i}\right) \cdot (4d + L_i - l) + \left(\frac{L_i - l}{L_i}\right) \cdot l = \left(\frac{l}{L_i}\right) \cdot (4d + 2(L_i - l)) \leq l \left(4 \left(\frac{d}{L_i}\right) + 2\right).$$
Since the composition is slack, we have $d/L_i \leq 1/2$, and hence the expression above is at most $4L_i$ as required.

While it might be the case that an outerplanar graph $G$ does not have a slack composition, we now show that $G$ can always be converted into a graph $H$ which does have a slack composition, at the cost of a small distortion.

**Lemma 5.3** Given an outerplanar graph $G = (V, E)$, there is an outerplanar graph $H = (V, E')$ (in fact, a subgraph of $G$) with a slack composition such that $d_G \geq d_H \geq \frac{1}{2} d_G$.

**Proof:** The graph $H$ will be a subgraph of $G$, with edge lengths no longer than in $G$ and no shorter than half those in $G$. Let $\{G_0 = P_0, P_1, \ldots, P_i\}$ be the composition defining $G$. Our goal is to produce a slack composition $(H_0 = Q_0, Q_1, \ldots, Q_f)$ for $H$, thereby defining $H$ in the process.

The composition sequence for $H$ is initially set to be the same as that for $G$; we then consider the lowest unmarked path $Q_i$, and while processing and marking the path $Q_i$, we modify possibly both the preceding (marked) and forthcoming (unmarked) paths. We maintain the following invariants during this process: $H$ is always a connected spanning subgraph of $G$; at each stage, the distances may only decrease; finally, the edge lengths never decrease by more than a factor of 2 from their original values.

To begin, $Q_0$ is marked. For each $i > 0$, if the path $Q_i$ is attached to a single vertex, we mark it and go on. Otherwise, $Q_i$ is attached to some edge $e_i = (u_i, v_i)$ lying on some $Q_k$ with $0 \leq k < i$. If $Q_i$ is slack at this point, we again mark it and continue. So assume that the current length of $Q_i$ is less than twice the current length of the edge $e_i = (u_i, v_i)$. We then do the following:

1. **Modify $Q_i$:** Decrease the lengths of all the edges in $Q_i$ by a factor of $1 \leq \text{length}(Q_i)/\text{length}(e_i) < 2$, so that the current length of $Q_i$ becomes exactly the current length of $e_i$. Remove $Q_i$ from the sequence for $H$. Note that the lengths of edges in $Q_i$ are halved in the worst case. They will never be changed again (except that the edges may possibly be removed later).

2. **Modify $Q_k$:** Recall that $Q_i$ was attached to the ends of $e_i$ lying on some previously marked path $Q_k$ with $k < i$. Since now $\text{length}(e_i) = \text{length}(Q_i)$, replace $e_i$ in $Q_k$ by the entire rescaled path $Q_i$ to get $Q'_k$. This does not change any current distances in the graph.

3. **Modify $Q_j$, $j > i$:** Observe that shrinking the path $Q_i$ may have resulted in some edges being longer than the current distance between their endpoints in the forthcoming (but not the preceding) paths. To overcome this problem, consider any such edge $e \in Q_j$. If there is a path $Q_{j'}$, with $j' > j$, that is attached to the endpoints of $e$ (and there can be only one such path), replace $e$ in $Q_j$ with $Q_{j'}$ and remove $Q_{j'}$ from the sequence. If there is no such $Q_{j'}$, deleting $e$ splits $Q_j$ into two paths, each attached to a single point, and we replace the old $Q_j$ in the composition with these two new paths. Again, note that this does not alter any current distances. We do not mark any paths in this modification.

The main properties of the above procedure are as follows. At each time step, we have connected spanning subgraphs of $G$. The edges surviving upon termination were modified at most once, and their lengths were decreased at that time by at most a factor of 2. No edge-length (and hence no
distance between any pair of vertices) is ever increased. The final sequence is slack. The process terminates when we have marked all the paths, i.e., in at most \(|E|\) steps.

Let \(H\) be the graph specified by the resulting slack sequence. It is a connected spanning subgraph of \(G\), with edge lengths at least half those in \(G\). This immediately implies the lower bound \(d_H \geq \frac{1}{2}d_G\). The upper bound \(d_H \leq d_G\) follows from the fact that none of the steps above caused distances to increase.

Now the overall procedure for embedding an outerplanar graph \(G\) is as follows. First, we obtain the graph \(H\) with a slack composition as in Lemma 5.3, incurring a distortion of at most 2. The graph \(H\) (with the edge lengths doubled in order to dominate \(G\)) is then embedded into a tree distribution with distortion at most 4 using Lemma 5.2, giving a total distortion of at most 8.

Furthermore, note that all the trees in the distribution are dominating subtrees of \(H\) with doubled edge lengths, and thus also dominating subtrees of \(G\). For each such tree \(T\), restoring the length of an edge \(e \in T\) to \(d_G(e)\) can only decrease the distortion without changing the domination property. Hence we get the main result of this section:

**Theorem 5.4** For any metric \(d_G\) supported on a \(K_{2,3}\)-free graph \(G\), there is an embedding of \(d_G\) into a tree distribution \(D\) with distortion at most 8. Moreover, the embedding uses only subtrees of \(G\) with their original edge lengths.

### 5.2 Tree embeddings for graphs with few edges

**Theorem 5.5** Any graph \(G\) with Euler characteristic \(\chi(G)\) can be embedded into a dominating tree distribution with distortion \(O(\log \chi(G) \log \log \chi(G))\).

**Proof:** The proof is very similar to that of Theorem 4.8. Recall that an isolated path in \(G\) is a path with endpoints of degree \(\geq 3\), and all internal nodes of degree 2. For every isolated path \(B = (v_1, v_2, \ldots, v_k)\) in \(G\), we add to \(G\) a new edge \(e_B\) between the endpoints of \(B\), of length \(d_G(v_1, v_k)\), thus leaving the original metric unaffected. Now, for each such \(B\), independently of other isolated paths, choose an edge \(e\) in \(B\) with probability proportional to the length of \(e\), and delete it. We get a distribution over graphs \(G'\), where each \(G'\) consists of the same “core” (including all the newly added edges), and the “hairs” (the remnants of the isolated paths).

Each \(G'\) dominates \(G\), and the expected expansion of any edge in \(B\) introduced by the above step is at most 2 (by an analysis very similar to (5.12)), implying that the distortion incurred by this distribution over \(G'\)-metrics is at most 2.

Finally, we have to embed each \(G'\) into a dominating tree distribution. It suffices to embed the core, since each hair is already a tree and can simply be attached to the random tree approximating the core. As in the proof of Theorem 4.8, we conclude that the number of vertices in the core is \(O(\chi(G))\), and hence it can be embedded into a distribution over trees with distortion \(O(\log \chi(G) \log \log \chi(G))\) by the general result of [5]. This completes the proof. ■
5.3 Lower bounds for series-parallel graphs

In view of the results of the previous sections, Theorems 5.4 and 5.5 may inspire hope that embeddings into tree distributions with constant distortion exist for other minor-closed families, such as series-parallel graphs. Our next result shows that this is not so; we prove a lower bound of $\Omega(\log n)$ on the distortion for embedding series-parallel graphs into dominating tree distributions. This result extends those of Alon et al. [1] and Konjevod et al. [24], who gave a technically more involved lower bound for the $n$-vertex grid, and shows that approximating graph metrics by distributions over tree metrics already breaks down for families of graphs that are much simpler than grids.

**Theorem 5.6** There exists an infinite family of series-parallel graphs $\{G_k\}$ such that any $\alpha$-approximation of the shortest-path metric of $G_k$ by a distribution over dominating trees has $\alpha = \Omega(\log |V(G_k)|)$.

The proof makes use of the following fact from [31]:

**Theorem 5.7 ([31])** The distortion of any embedding of the unit-weighted cycle $C_n$ into an (arbitrary) tree is at least $n/3 - 1$.

**Proof of Theorem 5.6:** The graphs $G_k$ are defined recursively. $G_0$ is a single unit-weighted edge between terminals $s_0$ and $t_0$. Inductively, $H_{i+1}$ consists of two copies of $G_i$ in series, and $G_{i+1}$ consist of two copies of $H_{i+1}$ in parallel between terminals $s_{i+1}$ and $t_{i+1}$ (see Figure 5.3). The graph $G_k$ has $n = 4^k$ edges and $\Theta(n)$ vertices. Observe that for any $G_i$ with terminals $s_i$ and $t_i$, both the distance between the terminals and the size of a minimum $s_i$-$t_i$ cut are $2^i$.

Following a standard framework for establishing lower bounds for probabilistic constructions (see, e.g., [37, 1, 24]), it suffices to come up with a distribution $D$ over the edges of $G_k$, such that any tree $T$ with $V(G_k) \subseteq V(T)$ and $d_T \geq d_{G_k}$ has a large expected expansion, i.e., $E_{e \in D}[d_T(u_e, v_e)] \geq \Omega(\log |V(G_k)|)$, where $u_e, v_e$ denote the endpoints of edge $e$. More concretely, it suffices to show that for any tree metric $d_T \geq d_{G_k}$ on $V(G)$ we have

$$\sum_{e \in E(G_k)} d_T(u_e, v_e) = \Omega(k) \cdot \sum_{e \in E(G_k)} d_{G_k}(u_e, v_e) = \Omega(k) \cdot 4^k,$$

since then the same must also hold for any distribution over dominating tree metrics, implying an expansion of $\Omega(k) = \Omega(\log |V(G_k)|)$.

Let $T$ be a tree containing the vertices of $G_k$ which dominates distances in $G_k$. For each $i \in [1, \ldots, k]$, assign color $i$ to all edges of $G_k$ which suffer an expansion of at least $2^{i+1}/3 \cdot 1$ in $T$. As a result, each edge in $G_k$ has at least one color assigned to it, while some edges have multiple colors. Let $S_i \subseteq E(G_k)$ be the set of all edges that are assigned color $i$.

How large is $S_k$? Observe that any cycle which goes around the graph $G_k$ (i.e., a simple cycle which includes the terminals $s_k$ and $t_k$) has length $2^{k+1}$, and therefore, by Theorem 5.7, contains an edge colored $k$. Thus $S_k$ hits all such cycles, and consequently it must separate the terminals of at least one of the four copies of $G_{k-1}$ that form $G_k$. Hence $|S_k| \geq 2^{k-1}$.
Figure 5.3: The graph $G_3$

How large is $S_{k-1}$? Consider the four copies of $G_{k-1}$ forming $G_k$. Arguing as before, we conclude that each of these copies must contain at least $2^{k-2}$ edges of color $k - 1$. Hence, the size of $S_{k-1}$ is at least $4 \cdot 2^{k-2}$. Arguing in the same vein for each $i$, we get that $|S_i| \geq 4^{k-i}2^{i-1} = 2^{2k-1-i}$.

For each $e \in E(G_k)$, let $C_e$ be the set of colors assigned to $e$. The expansion of $e$ is at least

$$\max_{i \in \mathcal{L}_e} \left( \frac{2^{i+1}}{3} - 1 \right) \geq \frac{1}{2} \sum_{i \in \mathcal{C}_e} \left( \frac{2^{i+1}}{3} - 1 \right).$$

Therefore,

$$\sum_{e \in E(G_k)} d_T(u_e, v_e) \geq \frac{1}{2} \sum_{i \in \mathcal{C}_e} \left( \frac{2^{i+1}}{3} - 1 \right) = \frac{1}{2} \sum_{i=1}^{k} |\{ i \in C_e \}| \cdot \left( \frac{2^{i+1}}{3} - 1 \right) =$$

$$\geq \frac{1}{2} \sum_{i=1}^{k} |S_i| \cdot \left( \frac{2^{i+1}}{3} - 1 \right) \geq \frac{1}{2} \sum_{i=1}^{k} 2^{2k-i-1} \cdot \left( \frac{2^{i+1}}{3} - 1 \right) > \left( \frac{k}{5} - \frac{1}{4} \right) 4^k.$$

\[ \blacksquare \]

Remark 5.8 After the preliminary version of this paper appeared, we were informed by Yair Bartal that Theorem 5.6 for the same family of graphs can also be inferred — albeit much less directly — from the result of Imase and Waxman [20] combined with the general framework of Bartal [4]. To see this, note that the Steiner tree problem is trivially 1-competitive on trees, and hence an $\alpha$-probabilistic approximation of $G_k$ by trees implies an $\alpha$-competitive ratio on the graphs $G_k$ [4, Theorem 4]. However, [20] establishes an $\Omega(k)$ lower bound for the competitive ratio for the Steiner problem on $G_k$, and hence $\alpha = \Omega(k)$.

5.4 An alternative embedding for series-parallel graphs

In light of the lower bound of the previous section, we cannot hope to embed general series-parallel graphs into tree distributions with constant distortion. However, by adding an extra ingredient (specifically, a cut-metric embedding of certain special series-parallel graphs which we call “bundles”) to the tree metric technology, we will be able to come up with an alternative
embedding of series-parallel graphs into $\ell_1$ with constant distortion which is quite different from that of Section 4.1.

The new embedding proceeds along the same lines as the embedding of outerplanar graphs in Section 5.1. Given a series-parallel graph $G$, it first performs preprocessing and random edge deletion steps similar to those in Lemmas 5.3 and 5.2 to get a special tree-like series-parallel graph which we call a “tree of bundles” (i.e., a graph whose 2-connected components are bundles). This incurs a distortion of at most 8. The bundles are then embedded using the cut-metric technique with distortion 2, yielding an embedding with total distortion at most 16 for general series-parallel graphs. Although it has a marginally worse performance guarantee (at least in terms of the constant bounds we have established here), this second algorithm is conceptually simpler, and arguably more instructive than that of Theorem 4.1. Since much of the construction is similar to that for outerplanar graphs given in Section 5.1, we shall omit the recurring details and emphasize the differences.

As in Section 4.1, the construction is based on the composition procedure for $G$. The compositions allowed here are slightly less restrictive than before, in that we add paths of arbitrary lengths between the ends of some existing edge at each stage, rather than a single vertex (i.e., a path of length 2). Hence the composition consists of a sequence of graphs $G_i$, where $G_0 = P_0$ is a path, and $G_i$ is obtained by attaching a path $P_i$ to already existing edge $e_i = (u_i, v_i)$. We require that the length of $P_i$ be no less than the length of $e_i$, and that the lengths of all edges are equal to the actual distance between their endpoints in $G$. We shall further relax the composition by permitting $P_i$ to be attached to just a single vertex; such a path will be called free.

Call a (non-free) path slack if its length $L_i$ is at least twice $d_i$, the length of the edge $e_i = (u_i, v_i)$. Similarly, a path is called taut if $L_i = d_i$. (Note that it is possible for a path to be neither taut nor slack.) We say a composition is slack-taut if each (non-free) path is either slack or taut. The first observation is that we can define a preprocessing step similar to that in Lemma 5.3 for series-parallel graphs, which outputs a series-parallel graph with a slack-taut composition.

**Lemma 5.9** Given a 2-connected series-parallel graph $G = (V, E)$, there is a series-parallel graph $H = (V, E')$ with a slack-taut composition such that $d_G \geq d_H \geq \frac{1}{2}d_G$.

The construction of $H$ and the proof of its correctness are very similar to those of Lemma 5.3. One small difference is that whenever we reduced the length of $P_i$ in the sequence defining an outerplanar graph, we could always remove the edge $(u_i, v_i)$ to which $P_i$ was attached. For series-parallel graphs, many paths can be attached to the same edge, so we cannot remove it. However, since the reduced path $P_i$ is taut, leaving $e_i$ in place satisfies the slack-tautness condition. Another small difference is that now we cannot remove a (forthcoming) edge which has become longer than the actual distance between its endpoints: this could contradict the technical requirement that paths must be attached to edges. To overcome this difficulty, we do not actually remove such an edge, but only mark it as “to be removed” and never touch it again until the end; then it is removed.

Before stating the next lemma, let us formally define a bundle as a series-parallel graph such that all simple paths between its terminals are of the same length. Note that a bundle has a well-defined length, which is the distance between its terminals. Figure 5.4 shows an example of a bundle with terminals $s$ and $t$. 

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Figure 5.4: A bundle: all non-labeled edges have unit length.

Consider the slack-taut composition of $H$ in Lemma 5.9. Observe that if $P_j$ is a taut path attached to a preceding path $P_i$, and $P_i$ is part of a bundle, then $P_j$ also becomes a part of the same bundle. In this way we obtain the maximal bundles of the graph $H$. Note that if a maximal bundle $B'$ is attached to two vertices on some other maximal bundle $B$ (and in particular, $B'$ cannot be considered a sub-bundle of $B$), then $B'$ must be at least twice as long as the distance between its terminals. This view allows us to define another slack composition for $H$, in which we attach slack (maximal) bundles at each step (instead of adding slack paths).

**Lemma 5.10** Given a series-parallel graph $H$ and a slack-taut composition for it, $H$ can be embedded into a distribution over special subgraphs with distortion at most $4$. The special subgraphs in this distribution have the property that all their maximal 2-connected components are bundles.

The proof is similar to that of Lemma 5.2. Consider the slack composition, where a slack bundle is attached at each step. This is analogous to the slack composition for outerplanar graphs, and we shall use it in a similar way. Specifically, when adding a bundle of length $L$, we choose a value $r \in [0, L]$ uniformly at random and cut all the edges that cross a point at distance $r$ from a fixed terminal of the bundle. The analysis of edge expansion is identical to that in the proof of Lemma 5.2. Since by cutting a bundle we create smaller bundles and some free paths, we obtain a “tree of bundles” at the end of the procedure.

The final step of the embedding has no outerplanar analog. Notice that bundles are precisely the special series-parallel graphs discussed in Lemma 4.4. Thus they can be embedded into $\ell_1$ with distortion at most 2 using the cut-metric technique.

Combining Lemmas 4.4, 5.9, and 5.10, we arrive at the main result of this section:

**Theorem 5.11** The procedure described in this section produces an embedding of series-parallel graphs into $\ell_1$ with distortion at most 16.

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**References**


A Appendix: Proof of equation (3.2)

Equation (3.2) follows from a general result concerning positive real vectors. Let $v, u \in \mathbb{R}^k$ be two positive vectors. Define

$$H(v, u) = \max_{i} \frac{v_i}{u_i} \cdot \max_{j} \frac{v_j}{u_j}.$$ 

If $S \subseteq \mathbb{R}^k$ is a closed set of positive vectors, define $H(v, S)$ as $\min_{u \in S} H(v, u)$.

Claim A.1 If $K \subseteq \mathbb{R}^k$ is a closed convex cone, then

$$H(v, K) = \max_{(C, D)} \frac{D \cdot v}{C \cdot v},$$

where the maximum is taken over all non-negative vectors $D, C \in \mathbb{R}^k$ for which $\frac{D_u}{C_u} \leq 1$ for any $u \in K$.

In the sequel, we use $\xi(v, K)$ to refer to the expression on the right hand side of (A.1). Before we prove Claim A.1, let us explain how it implies (3.2).

A metric $(V, \mu)$ on $|V| = n$ points can be viewed as a positive vector in $\mathbb{R}^3$, in which the value of the $ij$-th coordinate (for $i < j$) is $\mu(i, j)$. Since the set of $l_1$-embeddable metrics on a set $V$
coincides with the set of non-negative combinations of cut metrics on \( V \), they form a closed convex cone in \( \mathbb{R}^{\binom{V}{2}} \), called the cut cone (see, e.g., [15] for more details). Denote the cut cone on \( V \) by \( M_1(V) \).

Note that if \( v_\mu \) is the vector corresponding to a metric \((V, \mu)\), then \( H(v_\mu, M_1(V)) = c_1(\mu) \). Therefore, applying Claim A.1 to \( K = M_1(V) \) and \( v = v_\mu \), we obtain (3.2).

**Proof of Claim A.1:** One direction of the claim is easy: for any \( u \in K \) and \( D, C \) as above,

\[
\frac{D \cdot v}{C \cdot u} \leq \max_i \frac{u_i}{v_i} \cdot \max_j \frac{v_j}{u_j} \cdot \frac{D \cdot u}{C \cdot u} \leq H(v, u).
\]

Taking the “closest” \( u \in K \) to \( v \), we conclude that \( \xi(v, K) \leq H(v, K) \).

For the other direction, let \( B_\delta(v) \subseteq \mathbb{R}^k \) be the set of all positive vectors \( x \in \mathbb{R}^k \) such that \( H(v, x) \leq \delta \). Clearly,

\[
B_\delta(v) = \{ x \in \mathbb{R}^k \mid \forall r, q \in [1..k], \delta \cdot u_r x_q - v_q x_r \geq 0 \}.
\]

Observe that \( B_\delta(v) \) is a closed convex cone containing \( v \). By definition, \( H(v, K) \) is the smallest \( \delta \) such that \( B_\delta(v) \cap K \neq \emptyset \). For this critical \( \delta \), we claim that there exists a vector \( l \in \mathbb{R}^k \) such that

1. \( l \cdot B_\delta(v) \geq 0 \);
2. \( l \cdot K \leq 0 \);
3. \( l \) is a non-negative combination of vectors \( \Delta_{rq} \in \mathbb{R}^k \), \( r, q \in [1..k], r \neq q \), where \( \Delta_{rq} \) has \(-v_q \)
   in the \( r \)-th coordinate, \( \delta v_r \) in the \( q \)-th coordinate, and \( 0 \) in all other coordinates.

Indeed, the dual cone

\[
B^*_\delta = \{ y \in \mathbb{R}^k \mid \forall x \in B_\delta, \langle x, y \rangle \geq 0 \}
\]

is the convex hull of vectors \( \{ \Delta_{rq} \} \), and thus the normal vector to any supporting hyperplane of \( B_\delta(v) \) separating it from \( K \) has the required properties.

Let \( l^+ \) and \( l^- \) be two non-negative vectors in \( \mathbb{R}^k \) with \( l^+ - l^- = l \), formed by taking the positive and the negative coordinates of \( l \) respectively. By the first two properties of \( l \), for any \( u \in K \), \( \frac{l^+ u}{l^- u} \leq 1 \), while \( \frac{l^+ u}{l^- u} \geq 1 \). In the rest of the argument, \( l^+ \) will play the role of \( D \), while \( l^- \) will play the role of \( C \).

Given an arbitrary form \( \langle \sum_i d_i x_i \rangle / \langle \sum_i c_i x_i \rangle \) defined over non-negative \( x \in \mathbb{R}^k \) with non-negative coefficients \( d_i \) and \( c_i \), let us define a new form

\[
\left( \frac{\sum_i d_i x_i}{\sum_i c_i x_i} \right)^\# = \frac{\sum_i (d_i - \min(d_i, c_i)) x_i}{\sum_i (c_i - \min(d_i, c_i)) x_i}.
\]

Observe that if the value of the original form is \( \geq 1 \), then the value of the new form exceeds that of the old one. Using this observation and the fact that \( l = \sum \alpha_{rq} \Delta_{rq} \) for some non-negative \( \alpha_{rq} \)'s, we can infer that

\[
\xi(v, K) \geq \frac{l^+ \cdot u}{l^- \cdot u} = \left( \frac{\sum_{r} \alpha_{rq} \Delta_{rq}^+ \cdot v}{\sum_{r} \alpha_{rq} \Delta_{rq}^- \cdot v} \right)^\# \geq \frac{\sum_{r} \alpha_{rq} \Delta_{rq}^+ \cdot v}{\sum_{r} \alpha_{rq} \Delta_{rq}^- \cdot v} = \delta = H(v, K),
\]

\[ 30 \]
which establishes the claim.