1. Consider random walk on the following graph:

Asymptotically, the proportion of time that the walk spends at vertex $u$ is

- $\frac{1}{4}$
- $\frac{3}{10}$
- $\frac{1}{5}$
- $\frac{3}{5}$
- none of the above

2. Consider the Markov chain defined by the following transition matrix:

$$
\begin{pmatrix}
0 & 1/3 & 2/3 \\
1 & 0 & 0 \\
0 & 1/2 & 1/2
\end{pmatrix}
$$

The stationary probability of the third state (corresponding to the third row/column of the matrix) is

- $\frac{1}{3}$
- $\frac{1}{10}$
- $\frac{2}{5}$
- $\frac{3}{5}$
- none of the above

3. Suppose you are given a completely shuffled standard deck of 52 cards (i.e., each permutation is equally likely). Suppose you are able to peek at the bottom card and you see that it is the Ace of Hearts. Given this information, the variation distance of the distribution of the deck from the uniform distribution is

- $0$
- $\frac{1}{52!}$
- $\frac{1}{52}$
- $\frac{1}{52}$
- $\frac{51}{52}$
- $1$

[continued overleaf]
4. Consider the Markov chain on colorings discussed in Lecture 26. Recall that, in a given coloring \( X \), \( c \) is a “good” color for \( v \) if \( v \) can legally be colored with \( c \) in \( X \), else \( c \) is a “bad” color. Which of the following is a valid coupling between two copies \( (X_t), (Y_t) \) of this process? (In each case we specify only how \( X_t, Y_t \) each choose a vertex \( v \) and color \( c \); we then assume as usual that they color \( v \) with \( c \) if possible.) [Mark all that apply.]

- \( X_t \) and \( Y_t \) each choose vertices \( v, v' \) and colors \( c, c' \) independently.
- \( X_t, Y_t \) choose the same vertex \( v \) and choose independent colors \( c, c' \).
- \( X_t \) chooses a vertex \( v \) and color \( c \); \( Y_t \) chooses the same \( v \) and, if \( c \) is a good color for \( v \) in \( X_t \), then \( Y_t \) chooses a good color \( c' \) for \( v \) in \( Y_t \) u.a.r., else \( Y_t \) chooses a bad color for \( v \) in \( Y_t \) u.a.r.
- Let \( \pi \) be a fixed but arbitrary permutation of the set of colors. \( X_t, Y_t \) choose the same vertex \( v \); \( X_t \) chooses a color \( c \), and \( Y_t \) chooses the color \( c' = \pi(c) \).
- \( X_t \) chooses a vertex \( v \) and color \( c \); \( Y_t \) chooses a vertex \( v' \neq v \) u.a.r. and a color \( c' \neq c \) u.a.r.

One person pointed out that several of these schemes are unable to guarantee that, if \( X_t = Y_t \), then \( X_{t+1} = Y_{t+1} \). The intended meaning of the question was that the above scheme is used until \( X_t = Y_t \), after which we revert to the identity coupling (where \( X_t, Y_t \) make identical moves). Judging by the responses, it seems that most or all students understood the question this way. Apologies for any confusion.

5. Consider random walk on a connected bipartite graph \( G = (V, E) \). Which of the following statements are true? [Mark all that apply.]

- Since random walk on a bipartite graph is periodic, it does not have a stationary distribution.
- The distribution given by \( \pi(v) = \frac{d(v)}{2|E|} \), where \( d(v) \) is the degree of vertex \( v \), is stationary for the random walk.
- The stationary distribution of the random walk is unique.
- There exists a distribution \( \pi \) on \( V \) such that, for all \( x, y \in V \), the sequence of probabilities \( p_t^x(y) \rightarrow \pi(y) \) as \( t \rightarrow \infty \).

6. Suppose you are a dealer at a casino and have to make a decision on selecting one of the three strategies for shuffling a deck of cards discussed in the notes. Which of the following strategies would you use if you want to minimize the expected number of cards that move during the entire shuffling procedure?

- Random transposition shuffle
- Top-in-at-random shuffle
- Riffle shuffle
- Any of them (all are equivalent up to constant factors)