Quiz 2

Out: 7 Apr.; Due: 13 Apr.

Notes: Please submit your solutions on Gradescope by 5pm on Mon April 13. Your solution should consist of a copy of this quiz (both pages) with appropriate bubbles clearly shaded or marked. No justification or working is required. These questions should not take you much additional time if you have read and understood Notes 21 and 22.

1. Bella visits a casino and repeatedly places $1 bets on a fair(!) game, winning or losing $1 independently with probability $\frac{1}{2}$ each time. Let $(X_t)$ denote her capital at time $t$, with $X_0 = $100. Under which of the following stopping rules does the optional stopping theorem apply [mark all that apply]?

- Bella stops when she has won $200
- Bella stops when she has either won $200 or lost $500
- Bella stops when she has spent all her money
- Bella stops the last time her capital reaches $200
- Bella stops after making 200 bets

2. Let $(X_t)$ be a symmetric random walk on the integers, where the distribution of $X_{t+1}$ given $X_t$ is determined by

$$X_{t+1} = \begin{cases} 
X_t + 1 & \text{with probability } \frac{1}{6}; \\
X_t - 1 & \text{with probability } \frac{1}{6}; \\
X_t & \text{with probability } \frac{2}{3}.
\end{cases}$$

If $X_0 = 0$, then the expected minimum number $T$ of steps until $X_T = 10$ or $X_T = -5$ is

- 50
- 150
- 225
- 450

3. The main obstacle to applying the analysis of the randomized algorithm for 2-SAT in the notes to the same algorithm applied to MAX-2-SAT is

- there is no “reference” satisfying assignment $a^*$ to measure distance
- the process $(X_t)$ is no longer a supermartingale
- Max-2-SAT is NP-hard so we can’t expect to find a good approximation in polynomial time

4. In the proof of the ballot theorem, the claim that $(X_k)$ is a martingale relies on the following fact:

- $E[S_k|S_{k-1}] = \frac{k}{k-1} S_{k-1}$
- $E[S_k|S_{k-1}] = S_{k-1}$
- $E[S_{k-1}|S_k] = S_k$
- $E[S_{k-1}|S_k] = \frac{k-1}{k} S_k$

[continued overleaf]
5. Let \( \{E_i\} \) be a family of 100 events each of which occurs with probability at most \( \frac{1}{17} \), and each of which is mutually independent of all but at most 5 others. In the proof of the Lovász Local Lemma, the conditional probability that any one of the \( E_i \) occurs, given that any subset of the others does not occur, is at most [mark the one that best applies]:

- \( \frac{1}{17} \)
- \( \frac{1}{6} \)
- \( \frac{1}{e} \)
- \( (1 - \frac{1}{6})^{100} \)

6. Consider the Packet Routing in Networks problem from Lecture 22. In the described (non-constructive) upper bound, the total number of random waits each packet experiences is given by [select the tightest applicable bound]:

- \( 2^{O(\log^*(c+d))} \)
- \( O(\log d \cdot 2^{O(\log^*(c+d))}) \)
- \( O(d \cdot 2^{O(\log^*(c+d))}) \)
- \( O(d^2 \cdot 2^{O(\log^*(c+d))}) \)