**CS271 Randomness & Computation** **Fall 2022**

Lecture 21: November 3

*Instructor: Alistair Sinclair*

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## 21.1 The Optional Stopping Theorem

Let $(X_i)$ be a martingale with respect to a filter $(\mathcal{F}_i)$. Since $(X_i)$ is a martingale we have $\mathrm{E}[X_i] = \mathrm{E}[X_0]$ for all $i$. Our goal is to investigate when the above equality can be extended from a fixed time $i$ to a *random* time $T$. I.e., when can we claim that $\mathrm{E}[X_T] = \mathrm{E}[X_0]$ for a time $T$ that is a random variable (corresponding to some stopping rule)?

We first present an example to show that the equality is not always true for an arbitrary random time $T$. We then present the *optional stopping theorem (OST)*, which gives sufficient conditions under which the equality holds, and then give some simple applications.

**Example 21.1** *Consider a sequence of fair coin tosses and let $X_i = \#heads - \#tails$ of the first $i$ tosses. Then $(X_i)$ is a martingale and $\mathrm{E}[X_0] = 0$. Let $T$ be the first time such that $X_i \geq 17$, i.e., the first time the number of heads exceeds the number of tails by 17. We then have $\mathrm{E}[X_T] = 17 \neq \mathrm{E}[X_0]$. The key reason that the equality fails is because $\mathrm{E}[T] = \infty$.*

**Definition 21.2 (Stopping time)** *Let $(\mathcal{F}_i)$ be a filter. A random variable $T \in \{0, 1, 2, \dots\} \cup \{\infty\}$ is a* stopping time *for the filter $(\mathcal{F}_i)$ if the event $\{T = i\}$ is $\mathcal{F}_i$-measurable.*

This definition says that the event $\{T = i\}$ depends only on the history up to time $i$, i.e., there is no look-ahead. Observe that $T$ defined in Example 21.1 is a stopping time. However, if we let $T$ be the time of the last head before the first tail, then $T$ is not a stopping time because the event $T = i$ depends on what happens at time $i + 1$.

**Theorem 21.3 (Optional Stopping Theorem)** *Let $(X_i)$ be a martingale and $T$ be a stopping time with respect to a filter $(\mathcal{F}_i)$. Then $\mathrm{E}[X_T] = \mathrm{E}[X_0]$ provided the following conditions hold:*

1. $\Pr[T < \infty] = 1$.
2. $\mathrm{E}[|X_T|] < \infty$.
3. $\mathrm{E}[X_i I_{\{T>i\}}] \to 0$ *as* $i \to \infty$, *where* $I_{\{T>i\}}$ *is the indicator of the event* $\{T > i\}$.

The above set of conditions is among the weakest needed for the theorem to hold. For convenience, we note an alternative, stronger pair of conditions that is often more useful in practice. Namely, the Optional Stopping Theorem holds if

(i)$'$ $\mathrm{E}[T] < \infty$;

(ii)$'$ $\mathrm{E}[|X_i - X_{i-1}| \mid \mathcal{F}_{i-1}] \leq c$ for all $i$ and some constant $c$.

The proof of the Optional Stopping Theorem, along with several alternative sets of conditions, can be found in [GS01]. We now present some applications of the theorem.

## 21.2 Gambler's ruin

Consider the example of a gambler playing a fair game. The gambler starts with capital 0, and stakes \$1 in every round. He wins and loses with probability $\frac{1}{2}$ in each round. The gambler wins the game if he earns \$$b$ and is ruined if he loses \$$a$. The game ends when the gambler either wins or is ruined. We want to calculate the probability $p$ that the gambler is ruined.

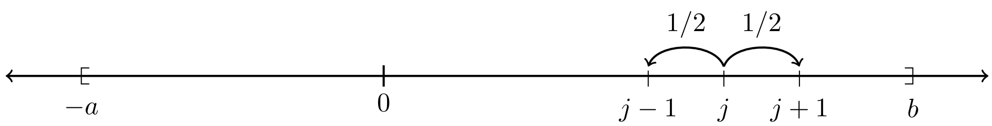


Figure 21.1: The gambling game starts at 0, and for any position $j$ goes to $j+1$ or $j-1$ with probability 1/2.

Let $X_i$ be the capital of the gambler at the end of round $i$. Then $X_i$ increases or decreases by 1 in every round with probability $\frac{1}{2}$. Hence $(X_i)$ is a martingale. Also, the martingale differences are clearly bounded in absolute value by 1, and moreover $\mathrm{E}[T] < \infty$[1], we see that conditions (i)$'$ and (ii)$'$ of the OST hold, and hence $\mathrm{E}[X_T] = \mathrm{E}[X_0]$. Thus we have

$$\mathrm{E}[X_T] = p \cdot (-a) + (1-p) \cdot b = \mathrm{E}[X_0] = 0 \qquad \Rightarrow \qquad p = \frac{b}{a+b}.$$

We can also use the Optional Stopping Theorem to estimate the expected duration of the game, i.e., E[# of steps before reaching $-a$ or $b$].

To this end, we define a new sequence of random variables $(Y_i)$ by $Y_i = X_i^2 - i$.

**Claim 21.4** *$(Y_i)$ is a martingale with respect to $(X_i)$.*

**Proof:**

$$\begin{aligned} \mathrm{E}[Y_i \mid X_1, \ldots, X_{i-1}] &= \mathrm{E}[X_i^2 - i \mid X_1, \ldots, X_{i-1}] = \tfrac{1}{2}\left((X_{i-1}+1)^2 - i\right) + \tfrac{1}{2}\left((X_{i-1}-1)^2 - i\right) = \\ &= (X_{i-1})^2 - (i-1) = Y_{i-1}. \end{aligned}$$

■

Let $T$ be the time when the player's balance reaches one of the game boundaries ($-a$ or $b$). We have shown that $\mathrm{E}[T] < \infty$, and we clearly also have $\mathrm{E}[|Y_i - Y_{i-1}| \mid X_0, \ldots, X_{i-1}] \leq 2\max\{a, b\} + 1$, which is bounded. Hence we can apply the Optional Stopping Theorem to the martingale $(Y_i)$ to obtain:

$$\mathrm{E}[Y_T] = \mathrm{E}[X_T^2] - \mathrm{E}[T] = \mathrm{E}[Y_0] = 0.$$

[1] To check this, for any integer $k \in [-a, b]$ let $T_k$ denote the expected time until the game ends starting with capital $X_0 = k$. Then $T_{-a} = T_b = 0$, and for $-a < k < b$ we have $T_k = 1 + \frac{1}{2} \cdot (T_{k-1} + T_{k+1})$. Clearly this set of difference equations has a finite solution.

But using our knowledge of the probabilities of terminating at $-a$ and at $b$, we may conclude that

$$\mathrm{E}[T] = \mathrm{E}[X_T^2] = a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = ab.$$

This is a strikingly simple proof of a non-trivial result.

## 21.3 Generalizations

We start by introducing a generalization of the concept of martingale.

**Definition 21.5** A stochastic process $(X_i)$ is a *submartingale* with respect to filter $(\mathcal{F}_i)$ if

$$\mathrm{E}[X_i \mid \mathcal{F}_{i-1}] \geq X_{i-1}.$$

It is a *supermartingale* if

$$\mathrm{E}[X_i \mid \mathcal{F}_{i-1}] \leq X_{i-1}.$$

Submartingales and supermartingales are useful extensions of the concept of martingale, and it can be proved that the Optional Stopping Theorem holds in these cases as well, with the corresponding inequality in the conclusion.

We now extend the analysis of the previous section to the case where the martingale difference $D_i = X_i - X_{i-1}$ may differ from $\pm 1$. Suppose all we know is that $(X_i)$ is a martingale, i.e., $\mathrm{E}[D_i \mid X_1, \ldots X_{i-1}] = 0$, and that the variance of the jump is bounded below, i.e.,

$$\mathrm{E}[D_i^2 \mid X_1, \ldots X_{i-1}] \geq \sigma^2.$$

The argument above generalizes if we pick $Y_i = X_i^2 - \sigma^2 \cdot i$. Then, we have:

$$\mathrm{E}[Y_i \mid X_1, \ldots, X_{i-1}] \geq Y_{i-1},$$

showing that $(Y_i)$ is a submartingale [**Exercise:** Check this!]. We can again apply the Optional Stopping Theorem to bound the expected length of the process:

$$\begin{aligned} \mathrm{E}[Y_T] &\geq \mathrm{E}[Y_0] \\ \mathrm{E}[X_T^2] - \sigma^2\mathrm{E}[T] = ab - \sigma^2\mathrm{E}[T] &\geq 0 \\ \mathrm{E}[T] &\leq \frac{ab}{\sigma^2}. \end{aligned}$$

This generalizes our earlier result; note that the inverse scaling by $\sigma^2$ (the second moment of the jumps) is natural: a process that makes only small jumps (small $\sigma$) will take a long time to exit the interval. [Note: When formally spelling out this kind of argument in practice, special provisions need to be made for the case when a jump lands outside the region $[-a, b]$. We shall ignore such details here.]

Let us now generalize further to the case where there is a drift in the walk. For variety, we will consider a slightly different scenario in which there is a reflecting barrier at one end of the interval, and we want to know how long it takes to reach the other end. Consider a supermartingale $(X_i)$, defined on the interval $[0, n]$ with $X_0 = s$. We assume the following:

$$\begin{aligned} \mathrm{E}[D_i \mid X_1, \ldots, X_{i-1}] &\leq 0 \\ \mathrm{E}[D_i^2 \mid X_1, \ldots, X_{i-1}] &\geq \sigma^2. \end{aligned}$$

The first condition is just the supermartingale property (i.e., a drift in the direction of 0); the second gives a lower bound on the jump sizes. Additionally, we assume that there is a reflecting barrier at the right-hand end of the interval, i.e., if $X_{i-1} = n$ then $X_i = n-1$ with probability 1. We are interested in $\mathrm{E}[T]$, where $T$ is the number of steps the walk takes to reach 0.

**Claim 21.6**

$$\mathrm{E}[T] \le \frac{2ns - s^2}{\sigma^2} \le \frac{n^2}{\sigma^2}.$$

**Proof:** Again, we define an auxiliary sequence of random variables $Y_i = X_i^2 + \lambda X_i + \mu i$. We will pick $\lambda$ and $\mu$ so that $\{Y_i\}$ is a submartingale. We have

$$\begin{aligned}
\mathrm{E}[Y_i \mid X_1, \ldots, X_{i-1}] &= \mathrm{E}[(X_{i-1} + D_i)^2 + \lambda(X_{i-1} + D_i) + \mu i \mid X_1, \ldots, X_{i-1}] = \\
&= X_{i-1}^2 + \lambda X_{i-1} + \mu i + (2X_{i-1} + \lambda) \cdot \mathrm{E}[D_i \mid X_1, \ldots, X_{i-1}] + \mathrm{E}[D_i^2 \mid X_1, \ldots, X_{i-1}] = \\
&= Y_{i-1} + (2X_{i-1} + \lambda) \cdot \mathrm{E}[D_i \mid X_1, \ldots, X_{i-1}] + (\mathrm{E}[D_i^2 \mid X_1, \ldots, X_{i-1}] + \mu).
\end{aligned}$$

By our assumptions on the differences $D_i$, this final expression will be bounded below by $Y_{i-1}$ provided we set $\mu = -\sigma^2$ and $\lambda = -2n$. Hence, with these values, $(Y_i)$ is a submartingale.

We can now apply the Optional Stopping Theorem to $(Y_i)$:

$$\begin{aligned}
\mathrm{E}[Y_T] &\ge \mathrm{E}[Y_0] \\
\mathrm{E}[X_T^2] - 2n\mathrm{E}[X_T] - \sigma^2 \mathrm{E}[T] &\ge s^2 - 2ns \\
\mathrm{E}[T] \le \frac{2ns - s^2}{\sigma^2} &\le \frac{n^2}{\sigma^2}
\end{aligned}$$

as $X_T^2 = X_T = 0$. It is not difficult to verify that the conditions for the application of the Optional Stopping Theorem hold in this case as well. [**Exercise:** Check this!] ■

Notice that the last bound is tight (even up to the constant factor) for a symmetric walk, for which $\sigma^2 = 1$ and $\mathrm{E}[T] = s(2n - s)$.

## 21.4 A simple algorithmic application: 2-SAT

It is well known that the 2-SAT problem can be solved in polynomial time (using strongly connected components in a directed graph). Here we shall see a very simple randomized polynomial time algorithm, due to Papadimitriou [P91] and independently McDiarmid [McD93], whose analysis makes use of the above results.

Here is the algorithm. Given a 2-CNF formula $\phi$ with $n$ variables, pick an arbitrary initial assignment $a_0$. Then in each round $i$, if $a_i$ denotes the current assignment and $a_i$ does not satisfy $\phi$, pick an arbitrary unsatisfied clause $C_i$. Choose a literal of $C_i$ uniformly at random and flip the value of that variable to obtain a new assignment $a_{i-1}$. Continue for $O(n^2)$ rounds.

**Claim 21.7** *If $\phi$ is satisfiable, then the above randomized algorithm finds a satisfying assignment w.h.p.*

**Proof:** Let $a^*$ be a satisfying assignment and let $X_i$ denote the Hamming distance between the assignment $a_i$ computed by the algorithm after $i$ rounds and $a^*$, i.e., the number of variables to which $a_i$ and $a^*$ assign different truth values. Then it is easy to see that, if $X_i > 0$:

$$\begin{aligned}
\Pr[|X_i - X_{i-1}| = 1] &= 1 \\
\Pr[X_i - X_{i-1} = -1] &\ge \frac{1}{2},
\end{aligned}$$

since at least one literal of $C_{i-1}$ has different values in $a_{i-1}$ and $a^*$. Letting $D_i = X_i - X_{i-1}$, we see that the process $(X_i)$ fits into the analysis above as long as $a_i$ does not satisfy $\phi$, as we have:

$$\begin{aligned} \mathrm{E}[D_i \mid X_1, \ldots, X_{i-1}] &\leq 0 \\ \mathrm{E}[D_i^2 \mid X_1, \ldots, X_{i-1}] &= \sigma^2 = 1. \end{aligned}$$

Now the number of rounds until a satisfying assignment is found is bounded above by the number of steps $t$ until $X_t$ reaches zero. And by the previous analysis this is bounded by

$$\mathrm{E}[\text{steps to reach } a^*] \leq \frac{n^2}{\sigma^2} = n^2.$$

(Note that a different satisfying assignment may in fact be found earlier than this, in which case the martingale analysis no longer applies; but that only makes things better for us, so the above upper bound on the time to find a satsifying assignment still holds.) ∎

**Open Problem:** Can the above ideas be used to obtain a simple constant-factor approximation algorithm for MAX-2-SAT? (Notice that the above analysis relies crucially on the existence of a "reference" satisfying assignment $a^*$.) Current (optimal) constant-factor approximation algorithms for MAX-2-SAT rely on heavier-duty machinery such as semi-definite programming [GW95].

## 21.5 The ballot theorem

In an election, suppose we have two candidates A and B, such that A receives more votes than B (let's say A receives $a$ votes, B recieves $b$ votes, and $a > b$). If votes are counted in random order, what is the probability that A remains ahead of B throughout the counting process? (For A to be "ahead," A's votes have to be strictly more than B's votes.)

The answer turns out to be $\frac{a-b}{a+b}$. This can be proved combinatorially, but there is a slick martingale proof which we now describe.

**Proof:** Let $S_k$ be (#A's votes) $-$ (#B's votes) after $k$ votes are counted; thus $S_n = a - b$, where $n = a + b$ is the total number of votes. Define $X_k = \frac{S_{n-k}}{n-k}$.

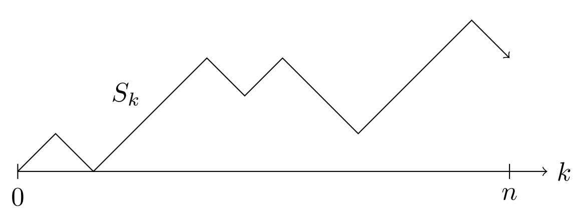


Figure 21.2: $S_k$ changes as vote counting unfolds. In this example, A is not always ahead of B as the path hits 0 after 2 steps

**Claim 21.8** *$(X_k)$ is a martingale.*

**Exercise:** Verify this claim! [Note: the martingale is defined "backwards" wrt the vote counting; it starts at $X_0 = \frac{S_n}{n}$.]

Let $T = \min\{k \mid X_k = 0\}$ or $T = n - 1$ if no such $k$ exists. There are two possibilities:

- Case 1: A is always ahead. Then $T = n - 1$, so $X_T = X_{n-1} = S_1 = 1$.
- Case 2: A is not always ahead. Then at some point in the process $X_k$ must be zero, which implies that $X_T = 0$.

Let $p$ be the probability that Case 1 occurs. Then $\mathrm{E}[X_T] = p \cdot 1 + (1-p) \cdot 0 = p$. By the Optional Stopping Theorem,

$$p = \mathrm{E}[X_T] = \mathrm{E}[X_0] = \mathrm{E}\left[\frac{S_n}{n}\right] = \frac{a-b}{a+b}.$$

■

The proof above is much simpler than standard combinatorial proofs based on the reflection principle.

## 21.6 Wald's equation

Let $\{X_i\}$ be i.i.d. random variables and $T$ a stopping time for $(X_i)$. Wald's equation says that the sum of $T$ of the $X_i$'s has expectation

$$\mathrm{E}[\sum_{i=1}^{T} X_i] = \mathrm{E}[T]\mathrm{E}[X_1],$$

provided $\mathrm{E}[T], \mathrm{E}[X_1] < \infty$. Note that we are summing a random number of the $X_i$'s.

**Proof:** This is left as an **exercise**. Show that, if $\mu = \mathrm{E}[X_i]$ is the common mean of the $X_i$, then

$$Y_i = \sum_{j=1}^{i} X_j - \mu i$$

is a martingale and use the Optional Stopping Theorem. To verify the conditions for the theorem, assume for simplicity that the $X_i$ are non-negative. ■

## 21.7 Percolation on $d$-regular graphs

As a final application of the Optional Stopping Theorem, we consider a result of Nachmias and Peres [NP10] concerning critical percolation on regular graphs.

In *p-percolation* on a graph $G = (V, E)$, we consider the random subgraph of $G$ obtained by including each edge of $G$ independently with probability $p$. When $G$ is the complete graph on $n$ vertices, this is nothing other than the Erdös-Rényi random graph model $\mathcal{G}_{n,p}$ that we have discussed earlier in the course. Recall that at the critical value $p = \frac{1}{n}$, the largest component in $\mathcal{G}_{n,p}$ is of size $O(n^{2/3})$ whp. Here we give a partial generalization of this result to the case of $d$-regular graphs for arbitrary $d$. ($\mathcal{G}_{n,p}$ is the case $d = n-1$.)

**Theorem 21.9** *Let $G$ be a $d$-regular graph on $n$ vertices, with $3 \le d \le n-1$, and let $\mathcal{C}_1$ be the largest component in $p$-percolation on $G$ with $p = \frac{1}{d-1}$. Then*

$$\Pr[|\mathcal{C}_1| \ge An^{2/3}] \le \frac{\alpha}{A^{3/2}},$$

*for some universal constant $\alpha$.*

Note that in the case of $\mathcal{G}_{n,p}$ the above probability is bounded much more sharply as $\exp(-\alpha' A^3)$.

The key ingredient in the proof of the above theorem is the following martingale lemma, which involves sophisticated use of Optional Stopping.

**Lemma 21.10** *Suppose $(X_t)$ is a martingale wrt a filter $(\mathcal{F}_t)$ and define the stopping time*

$$T_h = \min\{k, \min\{t : X_t = 0 \text{ or } X_t \geq h\}\}.$$

*Assume that*

*(i)* $X_0 = 1$ *and* $X_t \geq 0$ *for* $1 \leq t \leq k$*;*

*(ii)* $\mathrm{Var}[X_t \mid \mathcal{F}_{t-1}] \geq \sigma^2 > 0$ *when* $X_t > 0$*;*

*(iii)* $\mathrm{E}[X_{T_h}^2 \mid X_{T_h} \geq h] \leq Dh^2$ *for all* $h \geq \sqrt{\frac{\sigma^2 k}{D}}$*.*

*Then*

$$\Pr[X_t > 0 \ \ \forall t \leq k] \leq \frac{2\sqrt{D}}{\sigma\sqrt{k}}.$$

This lemma describes a martingale on the non-negative numbers with barriers at 0 and at $h$: the process stops when it hits (or exceeds) one of the barriers, or in any case after $k$ steps. The lemma bounds from above the probability that the process avoids hitting the barrier at 0.

**Proof of Lemma 21.10:** First, by the Optional Stopping Theorem[2] for $(X_t)$, we have

$$1 = \mathrm{E}[X_0] = \mathrm{E}[X_{T_h}] \geq h \Pr[X_{T_h} \geq h] \qquad \Rightarrow \qquad \Pr[X_{T_h} \geq h] \leq \frac{1}{h}. \tag{21.1}$$

Now define the auxiliary process $Y_t = X_t^2 - hX_t - \sigma^2 t$, for an arbitrary $h$ satisfying the lower bound in condition (iii). By the same kind of argument as earlier in this lecture, it is easy to check (using condition (ii) in the statement of the Lemma) that $(Y_t)$ is a submartingale as long as $X_t > 0$. Applying Optional Stopping to $(Y_t)$ we get

$$-h \leq \mathrm{E}[Y_0] \leq \mathrm{E}[Y_{T_h}] \leq (Dh^2 - h^2) \Pr[X_{T_h} \geq h] - \sigma^2 \mathrm{E}[T_h],$$

where in the last inequality we have used the second moment upper bound in (iii) and the fact that $X_t^2 - hX_t \leq 0$ for $0 \leq X_t \leq h$. Hence by (21.1) we have

$$\sigma^2 \mathrm{E}[T_h] \leq h + (Dh^2 - h^2) \cdot \frac{1}{h} = Dh$$

and thus $\mathrm{E}[T_h] \leq \frac{Dh}{\sigma^2}$. Markov's inequality now gives

$$\Pr[X_t > 0 \ \ \forall t \leq k] \leq \Pr[X_{T_h} \geq h] + \Pr[T_h \geq k] \leq \frac{1}{h} + \frac{Dh}{k\sigma^2}.$$

Finally, we optimize the bound by setting $h = \sqrt{\frac{\sigma^2 k}{D}}$ to get the result claimed in the lemma. ■

**Proof of Theorem 21.9:** Fix a vertex $v$ of $G$ and let $\mathcal{C}_G(v)$ denote the component containing $v$. Also, let $\mathcal{C}_T(v)$ denote the component containing $v$ in the $p$-percolation process on the infinite $d$-regular tree rooted at $v$. Clearly we can couple these two processes so that $|\mathcal{C}_G(v)| \leq |\mathcal{C}_T(v)|$.

[2] We omit the verification of the O.S.T. conditions in this proof; the reader should check these as an **exercise**!

To study $\mathcal{C}_T(v)$, we use essentially the same exploration process as we used in Lecture 16. Recall that this process performs a breadth-first search from $v$, maintaining at all times a list of "explored" vertices. Initially $v$ is the only explored vertex. At each step, we take the first remaining (unsaturated) explored vertex, mark all its unexplored neighbors as explored, and mark the vertex as saturated. The process dies when there are no remaining unsaturated explored vertices.

Let $X_t$ denote the number of unsaturated explored vertices at time $t$. Then $X_0 = 1$ and for $t \geq 1$, as long as $X_{t-1} > 0$ we have[3] $X_t = X_{t-1} - 1 + \mathrm{Bin}(d-1, \frac{1}{d-1})$. Note that $(X_t)$ is a martingale since the expectation of the binomial is 1. In order to apply Lemma 21.10, we need to compute the quantities $\sigma^2$ and $D$ as specified in the lemma. For $\sigma^2$ we need a lower bound on $\mathrm{Var}[X_t \mid \mathcal{F}_{t-1}]$, which is $\mathrm{Var}[\mathrm{Bin}(d-1, \frac{1}{d-1})] = (d-1)\frac{1}{d-1}(1 - \frac{1}{d-1}) = \frac{d-2}{d-1} \geq \frac{1}{2}$. So we may take $\sigma^2 = \frac{1}{2}$. For $D$ we need to bound $\mathrm{E}[X_{T_h}^2 \mid X_{T_h} \geq h]$, which we can do as follows using a r.v. $Z \sim \mathrm{Bin}(d-1, \frac{1}{d-1})$:

$$\begin{aligned} \mathrm{E}[X_{T_h}^2 \mid X_{T_h} \geq h] &\leq \mathrm{E}[(h+Z)^2] \\ &= h^2 + 2h\mathrm{E}[Z] + \mathrm{E}[Z^2] \\ &\leq h^2 + 2h + 1 \\ &\leq \frac{3}{2}h^2 \end{aligned}$$

for all $h \geq 5$. Hence we may take $D = \frac{3}{2}$ in Lemma 21.10. (Note that we will be applying the lemma with $k \sim n^{2/3}$, so certainly the condition $h \geq \sqrt{\frac{\sigma^2 k}{D}} \geq 5$ is satisfied.)

Lemma 21.10 with $\sigma^2 = \frac{1}{2}$ and $D = \frac{3}{2}$ now gives

$$\Pr[|\mathcal{C}_T(v)| \geq k] = \Pr[X_t > 0 \ \ \forall t \leq k] \leq \frac{2\sqrt{D}}{\sigma\sqrt{k}} = \frac{2\sqrt{3}}{\sqrt{k}} \leq \frac{4}{\sqrt{k}}.$$

Returning now to percolation on $G$ itself, let $N_k$ denote the number of vertices of $G$ contained in components of size at least $k$. Then

$$\mathrm{E}[N_k] = n \Pr[|\mathcal{C}_G(v)| \geq k] \leq n \Pr[|\mathcal{C}_T(v)| \geq k] \leq \frac{4n}{\sqrt{k}}.$$

Hence we have

$$\Pr[|\mathcal{C}_1| \geq k] \leq \Pr[N_k \geq k] \leq \frac{\mathrm{E}[N_k]}{k} \leq \frac{4n}{k^{3/2}}.$$

Finally, setting $k = An^{2/3}$ concludes the proof of the theorem. ■

# References

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[McD93] C.J.H. McDiarmid, "On a random recolouring method for graphs and hypergraphs," *Combinatorics, Probability and Computing* **2**, 1993, pp. 363–365.

[3] Technically this holds only for $t \geq 2$; for $t = 1$, $\mathrm{Bin}(d-1, \frac{1}{d-1})$ should be replaced by $\mathrm{Bin}(d, \frac{1}{d})$; we ignore this detail.

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[P91] C.H. Papadimitriou, "On selecting a satisfying truth assignment," *Proceedings of the 32nd IEEE FOCS*, 1991, pp. 163–169.