Homework 1 Solutions

Notes: These solutions are not necessarily model answers. Rather, they are designed to be tutorial in nature, and often contain more explanation than an ideal solution. Also, bear in mind that there may be more than one correct solution. The maximum total number of points available is 23.

1. (a) It is easier to calculate the probability of the complementary event, that the hand contains no ace. There are $\binom{52}{5}$ hands total, each having equal probability. The number of hands that contain no ace is $\binom{48}{5}$. Hence the probability that the hand contains no ace is *2pts*

$$\frac{\binom{48}{5}}{\binom{52}{5}} = \frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.66.$$

The probability that the hand contains an ace is just 1 minus this (i.e., about 0.34).

An alternative way to do the above calculation is as follows. For $i = 1, \ldots, 5$, let E_i denote the event that the ith card dealt is not an ace. Then the event that the hand contains no ace is exactly $\bigcap_{i=1}^{5} E_i$. This can be evaluated as follows:

$$\Pr[\bigcap_{i=1}^{5} E_i] = \Pr[E_1] \cdot \Pr[E_2|E_1] \cdots \Pr[E_5|\bigcap_{i=1}^{4} E_i] = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{44}{48}.$$

(b) Let E_1 be the event that the hand contains no card higher than a 10, and E_2 the event that the hand contains no card higher than a 9. The event we are interested in is precisely $E_1 \setminus E_2$, which has probability $\Pr[E_1] - \Pr[E_2]$. Each of these probabilities can be calculated in similar fashion to part (a). Thus *2pts*

$$\Pr[E_1] - \Pr[E_2] = \frac{36}{52} \cdot \frac{35}{51} \cdot \frac{34}{50} \cdot \frac{33}{49} \cdot \frac{32}{48} - \frac{32}{52} \cdot \frac{31}{51} \cdot \frac{30}{50} \cdot \frac{29}{49} \cdot \frac{28}{48} \approx 0.068.$$

An alternative approach is to compute $p_i = \Pr[E_2 \text{ and the hand contains exactly } i \text{ tens}] = \binom{4}{i}\binom{32}{5-i}/\binom{52}{5}$, and the probability we are interested in is given by $p_1 + p_2 + p_3 + p_4 + p_5$.

A third way is to use conditional probabilities, namely multiply $\Pr[\text{the hand contains at least one ten} \mid E_1]$ by $\Pr[E_1]$. The former is given by $1 - \Pr[\text{the hand contains no tens} \mid E_1] = 1 - \binom{32}{5}/\binom{36}{5}$.

A fourth way is to compute $1 - \Pr[\text{highest card is at most 9}] - \Pr[\text{highest card is at least jack}] = 1 - \Pr[E_2] - (1 - \Pr[E_1])$.

[NOTE: A potential pitfall here is to compute the probability using the expression $4 \times \binom{35}{4}/\binom{52}{5}$, based on the reasoning that there are 4 ways to pick a 10, and $\binom{35}{4}$ ways to pick 4 cards from the 36 cards that are at most 10, minus 1 to account for the first 10. This argument double-counts those hands with two or more 10s. A similar pitfall exists for part (a) as well.]

(c) The number of ways to pick a flush is $4\binom{13}{5}$ (four ways to choose the suit, then $\binom{13}{5}$ ways to choose five cards in that suit). The probability is thus $4\binom{13}{5}/\binom{52}{5}$, which simplifies to $\frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \approx 0.0020$. Equivalently, this same expression can be derived by observing that the first card can be anything, the second card must match the suit of the first (which happens with probability $\frac{12}{51}$), and so on. *2pts*

(d) There are 13 ways to choose the first value and 12 ways to choose the second value. Once these are chosen, there are $\binom{4}{3}$ ways to choose the three cards of the first value and $\binom{4}{2}$ ways to choose the two cards of the second value. The overall probability is thus $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}/\binom{52}{5} \approx 0.0014$. *2pts*

2. We use the principle of deferred decisions, imagining that the rolls on the first nine dice are fixed in advance and only the tenth die remains to be rolled. Let X_i denote the number of pips on the ith die, $i = 1, 2, \ldots, 10$, and let E_k denote the event that $X_1 + X_2 + \ldots + X_9 \equiv k \pmod 3$ for $k = 0, 1, 2$. Then, the probability that the sum of the pips on all 10 dice is divisible by 3 is given by *3pts*

$$\begin{aligned}
& \Pr[X_1 + X_2 + \ldots + X_{10} \text{ is divisible by 3}] \\
= \; & \sum_{k=0}^{2} \Pr[X_1 + X_2 + \ldots + X_{10} \text{ is divisible by 3} \mid E_k] \cdot \Pr[E_k] \\
= \; & \sum_{k=0}^{2} \Pr[X_{10} = 3 - k \text{ or } X_{10} = 6 - k \mid E_k] \cdot \Pr[E_k] \\
= \; & \sum_{k=0}^{2} \frac{1}{3} \cdot \Pr[E_k] \\
= \; & \frac{1}{3}(\Pr[E_0] + \Pr[E_1] + \Pr[E_2]) = \frac{1}{3}.
\end{aligned}$$

[NOTE*: The above argument does* **not** *assume that* $\Pr[E_1] = \Pr[E_2] = \Pr[E_3] = 1/3$*. It uses only the fact that these three probabilities sum to 1. It is in fact true that these three probabilities are all equal, but this requires an additional inductive argument to prove it. Some students did provide an inductive proof of this, which is fine (though rather more complicated than the above argument).*]

3. Let E denote the event that we observe a consecutive sequence of at least n heads. Let E_1 denote the event that the first n coins land heads, so $\Pr[E_1] = 2^{-n}$. For $i = 2, .., n+1$, let E_i denote the event that the $(i-1)$th coin lands tails, and the next n coins land heads, so $\Pr[E_i] = 2^{-(n+1)}$. Note that whenever the event E_i takes place, there are at most $i - 2 \le n - 1$ heads amongst the first $i - 2$ coins, so the *first* consecutive sequence of n heads must start from the ith coin. Hence, E_i is precisely the event that the first consecutive sequence of n heads start from the ith coin. Therefore, *3pts*

- $E = E_1 \cup E_2 \cup \cdots \cup E_{n+1}$; and
- the events $E_1, \ldots, E_{n+1}$ are mutually exclusive.

It follows that $\Pr[E] = \Pr[E_1] + \ldots + \Pr[E_{n+1}] = 2^{-n} + n2^{-(n+1)} = 2^{-n}(1 + n/2)$.

[NOTE*: A common mistake here was to forget to enforce a tail before the run of* n *heads: this leads to double-counting. Also, in the above proof, it is important to explain why the events* $E_1, E_2, \ldots, E_{n+1}$ *are mutually exclusive. Indeed, the argument used for this problem does not extend readily to computing the probability that we observe a consecutive sequence of at least* n *heads when we toss (say)* $3n$ *coins: make sure you understand why!*]

4. (a) Let E denote the event that a uniformly random ball is white, and W the event that all 10 balls are white. Our goal is to compute $\Pr[W \mid E]$. We do this using Bayes' rule: *2pts*

$$\Pr[W \mid E] = \frac{\Pr[E \mid W] \cdot \Pr[W]}{\Pr[E]}.$$

Clearly $\Pr[E \mid W] = 1$ and $\Pr[W] = \frac{1}{11}$. To compute $\Pr[E]$ we write

$$\begin{aligned}
\Pr[E] &= \sum_{i=0}^{10} \Pr[\text{there are exactly } i \text{ white balls}] \cdot \Pr[E \mid \text{there are exactly } i \text{ white balls}] \\
&= \sum_{i=0}^{10} \frac{1}{11} \cdot \frac{i}{10} = \frac{1}{2}.
\end{aligned}$$

Plugging all this into Bayes' rule, we get

$$\Pr[W \mid E] = \frac{1 \cdot \frac{1}{11}}{\frac{1}{2}} = \frac{2}{11}.$$

(b) Use the same notation as in part (a), except that now E denotes the event that all k balls chosen are white. The only probability that changes is $\Pr[E]$, which now becomes: *2pts*

$$\begin{aligned}\Pr[E] &= \sum_{i=0}^{10} \Pr[\text{there are exactly } i \text{ white balls}] \cdot \Pr[E \mid \text{there are exactly } i \text{ white balls}] \\ &= \sum_{i=0}^{10} \frac{1}{11} \cdot (\frac{i}{10})^k.\end{aligned}$$

Using Bayes' rule as before, we get

$$\Pr[W \mid E] = \frac{10^k}{1^k + 2^k + \cdots + 10^k}.$$

(Note that this probability tends rather quickly to 1 as k increases, as we would expect.)

5. (a) Let $C_1, \ldots, C_t$ denote all the distinct minimum cuts in the graph. From the analysis in class, the probability that the randomized min-cut algorithm outputs C_i is at least $\frac{2}{n(n-1)}$ for all $i = 1, 2, \ldots, t$, and these are mutually exclusive events. Hence, the probability that the randomized min-cut algorithm outputs some minimum cut is *1pt*

$$\sum_{i=1}^{t} \Pr[\text{the algorithm outputs } C_i] \geq \sum_{i=1}^{t} \frac{2}{n(n-1)} \geq \frac{2t}{n(n-1)}.$$

Since this probability can be at most 1, we see that $t \leq \frac{n(n-1)}{2}$, as required.

(b) Let G_n denote the graph comprising two $(n/2)$-cliques connected by a single edge if n is even, or a $\lfloor n/2 \rfloor$-clique and a $(\lfloor n/2 \rfloor + 1)$-clique connected by a single edge if n is odd. Note that the minimum cut of G_n has size 1. However, the size of the minimum cut increases to 2 whenever we merge two vertices from different cliques (to see why this is the case, observe that deleting any edge after we merge two vertices from different cliques still leaves a single connected component). So, if the modified algorithm terminates with a minimum cut, it must be the case that at *every* iteration (as long as we have at least 4 remaining vertices), it merges two vertices from one of the two cliques. In particular, this means that after each of the first $n/4$ iterations, the structure of the graph is still that of two cliques connected by a single edge. *3pts*

Letting E_i denote the event that in the ith iteration we choose two vertices from the same clique, we thus see that the probability of outputting a minimum cut is at most

$$\prod_{i=1}^{n/4} \Pr[E_i \mid \bigcup_{j=1}^{i-1} E_j]. \qquad (1)$$

To complete the analysis, we just need to bound the probability that the algorithm picks two vertices from the same clique, given that all previous picks have been of this form. Suppose that on some given iteration the smaller clique contains a fraction p of the remaining vertices (so the larger clique contains a fraction $1-p$). Then the probability of picking two vertices from the same clique is $p^2 + (1-p)^2$. This expression is maximized for p as close to 0 as possible. But since we are only considering the first $n/4$ iterations, the smaller clique must always have at least $n/2 - n/4 = n/4$ vertices, so $p \geq 1/3$ and the above probability is at most $(1/3)^2 + (2/3)^2 = 5/9$.

Hence every conditional probability in the expression (1) above is at most $5/9$, so the probability that the modified algorithm finds a minimum cut is at most $(5/9)^{n/4}$, which is $\leq c^{-n}$ for $c = (9/5)^{1/4}$.

[NOTES*: Quite a lot of students used the same example for G_n but falsely claimed that the probability that the cut survives each iteration is at most $\frac{1}{2}$. In fact it can get quite a bit larger than that (see above)–and indeed very large in the last few iterations, which is why we only consider $n/4$ iterations above.*

Also, as a very optional aside, it is interesting to observe that we may compute explicitly the probability the modified algorithm finds a minimum cut in G_n. For n even, say $n = 2k$, the probability is given by:

$$\frac{\binom{2k-2}{k-1} \cdot \left(\prod_{i=2}^{k} \binom{i}{2}\right)^2}{\prod_{i=3}^{2k} \binom{i}{2}}$$

The expression in the denominator corresponds to the number of ways to go about contracting each pair of vertices in G_n until we are left with 2 vertices. There is a $\prod_{i=2}^{k} \binom{i}{2}$ contribution to the numerator from each of the two n-cliques, and the $\binom{2k-2}{k-1}$ term comes from whether we're contracting an edge in the first clique or the second. We may rewrite this probability as

$$\frac{2\binom{2k-2}{k-1} \cdot (k!(k-1)!)^2}{(2k)!(2k-1)!} = \frac{2(k!)^2}{(2k)!(2k-1)} = \Theta(k^{-1/2}2^{-k})$$

using Stirling's approximation. The argument works because G_n has a unique minimum cut (which means the modified algorithm outputs a minimum cut if and only if it has contracted all the edges in each of the two cliques). Indeed, the same exponential lower bound applies to two cycles of almost equal size connected by a single edge. On the other hand, this argument does not apply to any pair of graphs on $n/2$ vertices connected by a single edge: e.g., take the graph with vertex set $\{1, 2, \ldots, n\}$ and edges $(n-1, n)$; $(i, n-1)$ for $i = 1, 2, \ldots, n/2$ and (i, n) for $i = n/2+1, \ldots, n-2$; this graph has many distinct minimum cuts of size 1.

Another potential example that in fact does not work is a clique on $n-1$ vertices connected to the n'th vertex by a single edge. As can be easily checked, the probability that the modified algorithm succeeds for this graph is $\prod_{k=3}^{n} \binom{k-1}{2} / \binom{k}{2} = \frac{2}{n(n-1)}$.]

(c) Let A_i denote the event that the algorithm outputs a minimum cut on the ith trial. If we perform T trials, then the probability that the algorithm outputs a minimum cut in some trial is given by *1pt*

$$\Pr[\bigcup_{i=1}^{T} A_i] \leq \sum_{i=1}^{T} \Pr[A_i] \leq Tc^{-n}.$$

Hence this probability certainly cannot be as large as $1/2$ as long as $T \leq c^n/2$. Thus we see that exponentially many trials are needed to reduce the probability of error to $1/2$.