Section 8

1. **(Probabilistic Method) (MU Exercise 6.3)** Given an n-vertex undirected graph $G = (V, E)$, consider the following method of generating an independent set. Given a permutaton σ of the vertices, define a subset $S(\sigma)$ of the vertices as follows: for each vertex i, $i \in S(\sigma)$ iff no neighbor j of i precedes i in the permutation σ.

   (a) Show that each $S(\sigma)$ is an independent set in G.

   (b) Suggest a natural randomized algorithm to produce σ for which you can show that the expected cardinality of $S(\sigma)$ is

   $$\sum_{i=1}^{n} \frac{1}{d_i + 1} \tag{1}$$

   where d_i denotes the degree of vertex i

   (c) Prove that G has an independent set of size at least $\sum_{i=1}^{n} 1/(d_i + 1)$

2. **(Probabilistic Method) (MU Exercise 6.10)** A family of subsets $\mathcal{F}$ of $\{1, ..., n\}$ is called an *antichain* if there is no pair of sets $A, B \in \mathcal{F}$ satisfying $A \subset B$.

   (a) Give an example of $\mathcal{F}$ where $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$.

   (b) Let f_k be the number of sets in $\mathcal{F}$ with size k. Show that

   $$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1 \tag{2}$$

   (Hint: Choose a random permutation of the numbers from 1 to n, and let $X_k = 1$ if the first k numbers in your permutation yield a set in $\mathcal{F}$. If $X = \sum_{k=0}^{n} X_k$, what can you say about X?)

   (c) Argue that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for any antichain $\mathcal{F}$.

3. **(Random Graphs) (MU Exercise 5.19)** An undirected graph on n vertices is *disconnected* if there exists a set of $k < n$ vertices such that there is no edge between this set and the rest of the graph. Otherwise, the graph is said to be *connected*. Show that there exists a constant c such that if $N \geq cn \log n$ then, with probability $1 - o(1)$, a graph randomly chosen from $G_{n,N}$ is connected.