1. Show that if a graph $G$ has $n$ vertices and $m$ edges, then there exists a cut of size at least $m/n + \frac{(n+1)}{2n}$.

[HINT: The uniform distribution over all partitions implies the existence of a cut of size at least $m/2$. Try a different distribution over partitions.]

2. In this problem, a 2-coloring of a graph $G$ is an assignment of colors red or green to the edges of $G$. Consider a random 2-coloring of $K_n$ (the complete graph on $n$ vertices), namely each edge is assigned a random color.

(a) Compute the expected number of monochromatic subgraphs $K_k$ in $K_n$ (for a fixed $k$ and $n$).

(b) Compute $n$ as a function of $k$ such that the expected number of monochromatic $K_k$ is 1 (the asymptotic value suffices). How about $k$ as a function of $n$?

We have seen in lecture that if $\binom{n}{k} \cdot 2^{1-(\frac{k}{2})} < 1$, then there exists a 2-coloring of $K_n$ so that it has no monochromatic $K_k$. Here, we generalize the result to handle the case $\binom{n}{k} \cdot 2^{1-(\frac{k}{2})} \geq 1$ (where we color a smaller graph $K_x$, $x \leq n$).

(c) Show that there exists a 2-coloring of $K_n$ such that the number of monochromatic $K_k$ is at most $\left\lfloor \binom{n}{k} \cdot 2^{1-(\frac{k}{2})} \right\rfloor$.

(d) Deduce that there exists a 2-coloring of $K_x$ so that it contains no monochromatic $K_k$, where

$$x = n - \left\lfloor \binom{n}{k} \cdot 2^{1-(\frac{k}{2})} \right\rfloor.$$

[NOTE: Note that $x = n$ whenever $\binom{n}{k} \cdot 2^{1-(\frac{k}{2})} < 1$.]