Section 6

1. **(Poisson Approximation) (Book Exercise 5.10)** Consider the probability that every bin receives exactly one ball when n balls are thrown randomly into n bins.

   (a) Give an upper bound on this probability using the Poisson approximation.

   (b) Determine the *exact* probability of this event.

   (c) Show that these two probabilities differ by a multiplicative factor that equals the probability that a Poisson random variable with parameter n takes on the value n. Explain why this is implied by Theorem 5.6.

2. **(Balls in Bins) (Book Exercise 5.11)** Consider throwing m balls into n bins, and for convenience let the bins be numbered from 0 to $n - 1$. We say there is a k-gap starting at bin i if bins $i, i + 1, ..., i + k - 1$ are all empty.

   (a) Determine the expected number of k-gaps.

   (b) Prove a Chernoff-like bound for the number of k-gaps. (Hint: If you let $X_i = 1$ when there is a k-gap starting at bin i, then there are dependencies between X_i and X_{i+1}; to avoid these dependencies, you might consider X_i and X_{i+k}.)

3. **(Balls in Bins) (Book Exercise 5.12)** The following problem models a simple distributed system wherein agents contend for resources but "back off" in the face of contention. Balls represent agents, and bins represent resources.

   The system evolves over rounds. Every round, balls are thrown independently and uniformly at random into n bins. Any ball that lands in a bin by itself is served and removed from consideration. The remaining balls are thrown again in the next round. We begin with n balls in the first round, and we finish when every ball is served.

   (a) If there are b balls at the start of a round, what is the expected number of balls at the start of the next round?

   (b) Suppose that every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in $O(\log \log n)$ rounds. (Hint: If x_j is the expected number of balls left after j rounds, show and use that $x_{j+1} \leq x_j^2/n$.)