Section 5

1. **(Poisson Random Variables) (Book Exercise 5.6)** Let X be a Poisson random variable with mean μ, representing the number of errors on a page of this book. Each error is independently a grammatical error with probability p and a spelling error with probability $1-p$. If Y and Z are random variables representing the number of grammatical and spelling errors (respectively) on a page of this book, prove that Y and Z are Poisson random variables with means μp and $\mu(1-p)$, respectively. Also, prove that Y and Z are independent.

2. **(Poisson Processes)** Consider a square-shaped portion of a city with four quadrants: restaurants, offices, apartments, and a park, placed as square portions from left to right then top to bottom. Suppose we want to model the number of Google searches made in different parts of the city at different times.

   (a) The offices have N laptops distributed roughly uniformly across the quadrant. In a 5 minute window near 10 AM on Tuesdays, each laptop makes a Google search with probability p_1. Approximate the distribution of the number of searches made in the offices from 9:45 to 10:15 AM on Tuesday.

   (b) Many office workers go to the restaurants or the park from 12-1 PM on Tuesdays, but still use Google on their phone. Suppose that the restaurants have $N/10$ phones in them from 11 AM to 11:30 AM, $N/6$ from 11:30 AM to 12 PM, and $N/4$ from 12 to 12:30 PM. In a 5 minute window throughout, each phone makes a Google search with probability p_2. Describe how to approximate the distribution of the number of searches made in the restaurants from 11 AM to 12:30 PM.

   (c) One portion of the power grid serves an irregularly shaped blob covering all four quadrants. Describe how to approximate the distribution of the number of missed Google searches if the power were to be lost in that blob for 30 minutes at 12 PM on a Tuesday. Assume the distribution of devices making Google searches is uniform across each quadrant.

   (d) How could you generalize these solutions for a continuous movement of workers or a non-uniform distribution of devices within each quadrant?

3. **(Chernoff Bounds) (Book Exercise 4.19)** Recall that a function f is said to be convex if, for any x_1, x_2 and for $0 \leq \lambda \leq 1$:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \tag{1}$$

   (a) Let Z be a random variable that takes on a (finite) set of values in the interval $[0,1]$, and let $p = \mathbf{E}[Z]$. Define the Bernoulli random variable X by $Pr(X=1) = p$ and $Pr(X=0) = 1-p$. Show that $E[f(Z)] \leq E[f(X)]$ for any convex function f.

   (b) Use the fact that $f(x) = e^{tx}$ is convex for any $t \geq 0$ to obtain a Chernoff bound for the sum of n independent random variables with distribution Z as in part (a), based on a Chernoff bound for independent Poisson trials.