1. (Chernoff) (Book 4.5) We plan to conduct an opinion poll to find out the percentage of people in a community who want its president impeached. Assume that every person answers yes or no. If the actual fraction of people who want the president impeached is $p$, we want to find an estimate $X$ of $p$ such that

$$\Pr(|X - p| \leq \varepsilon p) > 1 - \delta$$

for a given $\varepsilon$ and $\delta$, with $0 < \varepsilon, \delta < 1$.

We query $N$ people chosen independently and uniformly at random from the community and output the fraction of them who want the president impeached. How large should $N$ be for our result to be a suitable estimator of $p$? Use Chernoff bounds, and express $N$ in terms of $p, \varepsilon$, and $\delta$. Calculate the value of $N$ from your bound if $\varepsilon = 0.1$ and $\delta = 0.05$ and if you know that $p$ is between 0.2 and 0.8.

2. (Tail Inequalities) (Book Exercise 4.2) We have a standard six-sided die. Let $X$ be the number of times that a 6 occurs over $n$ throws of the die. Let $p$ be the probability of the event $X \geq n/4$. Compare the best upper bounds on $p$ that you can obtain using Markov’s inequality, Chebyshev’s inequality, and Chernoff bounds.

3. (Permutations and Chernoff) (Book Exercise 4.8) We show how to construct a random permutation $\pi$ on $[1, n]$, given a black box that outputs numbers independently and uniformly at random from $[1, k]$ where $k \geq n$. If we compute a function $f[1, n] \rightarrow [1, k]$ with $f(i) \neq f(j)$ for $i \neq j$, this yields a permutation; simply output the numbers $[1, n]$ according to the order of the $f(i)$ values. To construct such a function $f$, do the following for $j = 1, \ldots, n$: choose $f(j)$ by repeatedly obtaining numbers from the black box and setting $f(j)$ to the first number found such that $f(j) \neq f(i)$ for $i < j$. Prove that this approach gives a permutation chosen uniformly at random from all permutations. Find the expected number of calls to the black box that are needed when $k = n$ and $k = 2n$. For the case $k = 2n$, argue that the probability that each call to the black box assigns a value of $f(j)$ to some $j$ is at least 1/2. Based on this, use a Chernoff bound to bound the probability that the number of calls to the black box is at least $4n$.

4. (Chernoff) (Book Exercise 4.12) Consider a collection $X_1, \ldots, X_n$ of $n$ independent geometrically distributed random variables with mean 2. Let $X = \sum_{i=1}^{n} X_i$ and $\delta > 0$.

(a) Derive a bound on $\Pr(X \geq (1 + \delta)(2n))$ by applying a Chernoff bound to a sequence of $(1 + \delta)(2n)$ fair coin tosses.

(b) Derive a bound on $\Pr(X \geq (1 + \delta)(2n))$ using the moment generating function for geometric random variables.

(c) Which bound is better?