Section 3

1. **(Moments) (Book Exercise 3.12)** Find an example of a random variable with finite expectation and unbounded variance. Give a clear argument showing that your choice has these properties.

2. **(Tail Inequalities) (Book Exercise 3.16-3.17)** Show that the Markov and Chebyshev inequalities are tight (find random variables such that the inequalities are true with equality)

3. **(Tail Inequalities) (Book Exercise 3.21)**

   (a) Chebyshev's inequality uses the variance of a random variable to bound its deviation from its expectation. We can also use higher moments. Suppose that we have a random variable X and an even integer k for which $\mathbf{E}[(X - \mathbf{E}[X])^k]$ is finite. Show that

   $$\mathbb{P}(|X - \mathbf{E}[X]| > t(\mathbf{E}[(X - \mathbf{E}[X])^k])^{1/k})) \leq \frac{1}{t^k}$$

   (b) Why is it difficult to derive a similar inequality when k is odd?

4. **(Tail Inequalities) (Book Exercise 3.20)**

   (a) Let Y be a non-negative discrete random variable with positive expectation. Prove:

   $$\frac{\mathbf{E}[Y]^2}{\mathbf{E}[Y^2]} \leq \mathbb{P}(Y \neq 0)$$

   Hint: Recall that $(a+b)^2 \geq 2ab$.

   (b) Derive a lower bound on $\mathbb{P}(Y \neq 0)$ using Chebyshev's inequality. Compare this inequality to the inequality from part (a) and find a random variable where the bound from (a) is better. Hint: take $a = \mathbf{E}[Y]$ in Chebyshev's inequality.